Partial Differential Equations (First Order)

1. Formation of Partial Differential Equation

A partial differential equation (PDE) is an equation involving partial derivatives of a function of two or more independent variables. The formation of a PDE is done by eliminating arbitrary constants or arbitrary functions from a given relation.

Example: Form the PDE from $z = ax + by$, where a and b are constants.

Differentiate partially with respect to x: $\partial z/\partial x = a$
Differentiate partially with respect to y: $\partial z/\partial y = b$
Eliminating a and b, we get the PDE: $z = x(\partial z/\partial x) + y(\partial z/\partial y)$.

2. Classification of First Order Partial Differential Equations

A general first order PDE can be written as $F(x, y, z, p, q) = 0$, where $p = \partial z/\partial x$ and $q = \partial z/\partial y$.

Types:
1. Linear PDE: $ap + bq = c$, where a, b, c are functions of x and y.
2. Non-linear PDE: p and q appear non-linearly, e.g., $p^2 + q^2 = 1$.
3. Lagrange's Type: $Pp + Qq = R$.
4. Charpit's Type: $F(x, y, z, p, q) = 0$ (general non-linear form).

3. Lagrange's Method

Lagrange's method is used to solve first order linear PDEs of the form:
$Pp + Qq = R$.

The auxiliary equations are:
$dx/P = dy/Q = dz/R$.

Example: Solve $p + q = z$.

Here, $P = 1$, $Q = 1$, $R = z$.
Auxiliary equations: $dx/1 = dy/1 = dz/z$.
From $dx = dy \Rightarrow x - y =$ C■.
From $dx = dz/z \Rightarrow \ln z - x =$ C■.
Hence, the complete solution is $F(x - y, \ln z - x) = 0$.

4. Charpit's Method

Charpit's method is used to solve non-linear first order PDEs of the form:
$F(x, y, z, p, q) = 0$.

Charpit's auxiliary equations are:
$dx / F_p = dy / F_q = dz / (pF_p + qF_q) = dp / (-F_x - pF_z) = dq / (-F_y - qF_z)$.

Example: Solve $p^2 + q^2 = 1$.

Here, $F = p^2 + q^2 - 1$.
$F_p = 2p$, $F_q = 2q$, $F_x = F_y = F_z = 0$.

Thus, $dp = 0$ and $dq = 0 \Rightarrow p = a$, $q = b$ (constants).
Given $a^2 + b^2 = 1$.
Integrating: $dz = a\,dx + b\,dy$.
Hence, $z = ax + by + c$, where $a^2 + b^2 = 1$.