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Mathematical Methods

Topic: Orthogonalisation and Inner Product Spaces

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1 Inner Product Spaces

1.1 Definition and Axioms

Let V be a vector space over the field $\mathbb{F}$ (where $\mathbb{F}$ is $\mathbb{R}$ or $\mathbb{C}$). An **inner product** on V is a function $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{F}$ that satisfies the following axioms for all $u, v, w \in V$ and scalar $\alpha \in \mathbb{F}$:

1. **Conjugate Symmetry:** $\langle u, v \rangle = \overline{\langle v, u \rangle}$ (In $\mathbb{R}$, $\langle u, v \rangle = \langle v, u \rangle$).
2. **Linearity in the first argument:**

$$\langle \alpha u + v, w \rangle = \alpha \langle u, w \rangle + \langle v, w \rangle$$

3. **Positive Definiteness:** $\langle u, u \rangle \geq 0$, and $\langle u, u \rangle = 0$ if and only if $u = 0$.

A vector space equipped with an inner product is called an **Inner Product Space** (or Pre-Hilbert Space).

1.2 Norm and Metric

The inner product induces a norm on V:

$$\|u\| = \sqrt{\langle u, u \rangle} \tag{1}$$

This further induces a metric $d(u, v) = \|u - v\|$.

1.3 Function Spaces: $L^2[a, b]$

In the context of Mathematical Methods, we often deal with the space of square-integrable functions. Let $w(x)$ be a non-negative weight function on $[a, b]$. The inner product is defined as:

$$\langle f, g \rangle = \int_a^b f(x)\overline{g(x)}w(x)\,dx \tag{2}$$

Two functions f and g are **orthogonal** if $\langle f, g \rangle = 0$.

2 Orthonormal Sets

Definition 2.1 (Orthonormal Set). *A set of vectors $S = \{\phi_1, \phi_2, \dots\}$ in an inner product space V is called* ***orthonormal*** *if:*

$$\langle \phi_m, \phi_n \rangle = \delta_{mn} = \begin{cases} 1 & \text{if } m = n \\ 0 & \text{if } m \neq n \end{cases} \tag{3}$$

Theorem 2.1 (Linear Independence). *Every orthonormal set is linearly independent.*

Proof. Consider a finite linear combination $\sum_{i=1}^{k} c_i\phi_i = 0$. Taking the inner product with ϕ_j (where $1 \leq j \leq k$):

$$\langle\sum_{i=1}^{k} c_i\phi_i, \phi_j\rangle = \langle 0, \phi_j\rangle$$

$$\sum_{i=1}^{k} c_i\langle\phi_i, \phi_j\rangle = 0$$

Using orthonormality $\langle\phi_i, \phi_j\rangle = \delta_{ij}$:

$$c_j = 0$$

Since this holds for all j, all coefficients are zero. Thus, the set is linearly independent. □

3 The Gram-Schmidt Orthogonalisation Process

The Gram-Schmidt process is an algorithm for constructing an orthogonal basis from an arbitrary basis.

3.1 The Algorithm

Let $\{u_1, u_2, \ldots, u_n, \ldots\}$ be a linearly independent set of vectors in V. We construct an orthogonal set $\{v_1, v_2, \ldots\}$ and an orthonormal set $\{e_1, e_2, \ldots\}$ as follows:

Step 1: Set the first vector.

$$v_1 = u_1$$

$$e_1 = \frac{v_1}{\|v_1\|}$$

Step 2: Construct v_2 by subtracting the projection of u_2 onto e_1.

$$v_2 = u_2 - \langle u_2, e_1\rangle e_1$$

$$e_2 = \frac{v_2}{\|v_2\|}$$

Step k (General Step):

$$v_k = u_k - \sum_{j=1}^{k-1} \mathrm{proj}_{v_j} u_k = u_k - \sum_{j=1}^{k-1} \langle u_k, e_j\rangle e_j$$

$$e_k = \frac{v_k}{\|v_k\|}$$

The span of the new orthonormal set $\{e_1, \ldots, e_k\}$ is identical to the span of the original set $\{u_1, \ldots, u_k\}$.

4 Application: Construction of Legendre Polynomials

We apply the Gram-Schmidt process to the standard polynomial basis $S = \{1, x, x^2, x^3, \ldots\}$ on the interval $[-1, 1]$ with weight function $w(x) = 1$. The inner product is:

$$\langle f, g\rangle = \int_{-1}^{1} f(x)g(x)\, dx$$

4.1 Derivation

1. First Polynomial ($n = 0$): Let $u_0 = 1$.

$$\|u_0\|^2 = \int_{-1}^{1} 1 \cdot 1 \, dx = 2 \implies \|u_0\| = \sqrt{2}$$

Normalized: $e_0(x) = \frac{1}{\sqrt{2}}$. *(Note: Legendre Polynomials P_n are traditionally standardized such that $P_n(1) = 1$, not by unit norm. We will find the orthogonal set v_n first.)* Let $P_0(x) = 1$.

2. Second Polynomial ($n = 1$): Let $u_1 = x$. We remove the component along P_0.

$$v_1 = x - \frac{\langle x, P_0 \rangle}{\langle P_0, P_0 \rangle} P_0$$

Compute inner product: $\langle x, 1 \rangle = \int_{-1}^{1} x \, dx = 0$. Thus, $v_1 = x$. Let $P_1(x) = x$.

3. Third Polynomial ($n = 2$): Let $u_2 = x^2$.

$$v_2 = x^2 - \frac{\langle x^2, P_0 \rangle}{\langle P_0, P_0 \rangle} P_0 - \frac{\langle x^2, P_1 \rangle}{\langle P_1, P_1 \rangle} P_1$$

Compute terms:

- $\langle P_0, P_0 \rangle = 2$.
- $\langle x^2, P_0 \rangle = \int_{-1}^{1} x^2 \, dx = \left[\frac{x^3}{3} \right]_{-1}^{1} = \frac{2}{3}$.
- $\langle P_1, P_1 \rangle = \int_{-1}^{1} x^2 \, dx = \frac{2}{3}$.
- $\langle x^2, P_1 \rangle = \int_{-1}^{1} x^3 \, dx = 0$ (Odd function).

Substituting back:

$$v_2 = x^2 - \frac{2/3}{2}(1) - 0 = x^2 - \frac{1}{3}$$

Standardization: Legendre polynomials require $P_n(1) = 1$. $v_2(1) = 1 - 1/3 = 2/3$. We multiply by $3/2$.

$$P_2(x) = \frac{3}{2}\left(x^2 - \frac{1}{3}\right) = \frac{1}{2}(3x^2 - 1)$$

4. Fourth Polynomial ($n = 3$): Let $u_3 = x^3$.

$$v_3 = x^3 - \frac{\langle x^3, P_0 \rangle}{\|P_0\|^2} P_0 - \frac{\langle x^3, P_1 \rangle}{\|P_1\|^2} P_1 - \frac{\langle x^3, P_2 \rangle}{\|P_2\|^2} P_2$$

By symmetry (odd/even integrals), $\langle x^3, P_0 \rangle = 0$ and $\langle x^3, P_2 \rangle = 0$. We only need the P_1 term:

$$\langle x^3, P_1 \rangle = \int_{-1}^{1} x^4 \, dx = \frac{2}{5}$$

$$\|P_1\|^2 = \frac{2}{3}$$

$$v_3 = x^3 - \frac{2/5}{2/3} x = x^3 - \frac{3}{5} x$$

Standardizing for $P_3(1) = 1$:

$$P_3(x) = \frac{5}{2}\left(x^3 - \frac{3}{5}x\right) = \frac{1}{2}(5x^3 - 3x)$$

This process confirms the standard Legendre polynomials derived purely via orthogonalisation.

5 Approximation Theory

5.1 Least Squares Approximation

Let $f \in L^2[a, b]$. We wish to approximate f using a linear combination of orthonormal functions $\{e_1, \ldots, e_n\}$. Let $S_n = \sum_{k=1}^{n} c_k e_k$. We define the Mean Square Error E:

$$E = \|f - S_n\|^2 = \int_a^b \left| f(x) - \sum_{k=1}^{n} c_k e_k(x) \right|^2 w(x)\, dx$$

Theorem 5.1. *The error E is minimized when the coefficients c_k are the* ***Fourier Coefficients****:*

$$c_k = \langle f, e_k \rangle$$

Proof. Expand the norm:

$$E = \langle f - \sum c_k e_k, f - \sum c_j e_j \rangle$$

$$E = \langle f, f \rangle - \sum \overline{c_k} \langle f, e_k \rangle - \sum c_k \langle e_k, f \rangle + \sum \sum c_k \overline{c_j} \langle e_k, e_j \rangle$$

Using orthonormality $\langle e_k, e_j \rangle = \delta_{kj}$:

$$E = \|f\|^2 - \sum \overline{c_k} \langle f, e_k \rangle - \sum c_k \overline{\langle f, e_k \rangle} + \sum |c_k|^2$$

Let $\alpha_k = \langle f, e_k \rangle$. Then:

$$E = \|f\|^2 + \sum_{k=1}^{n} |c_k - \alpha_k|^2 - \sum_{k=1}^{n} |\alpha_k|^2$$

Since $|c_k - \alpha_k|^2 \geq 0$, E is minimized when $c_k = \alpha_k$. □

5.2 Bessel's Inequality

Since the minimum error $E_{min} \geq 0$:

$$\|f\|^2 - \sum_{k=1}^{n} |\langle f, e_k \rangle|^2 \geq 0$$

$$\sum_{k=1}^{n} |c_k|^2 \leq \|f\|^2 \tag{4}$$

This is Bessel's Inequality. It states that the sum of the energies of the components cannot exceed the total energy of the function.

5.3 Parseval's Identity and Completeness

If the set $\{e_n\}$ is **complete** (a basis for the entire Hilbert space), then as $n \to \infty$, the error approaches zero.

$$\sum_{n=1}^{\infty} |c_n|^2 = \|f\|^2 \tag{5}$$

This equality is Parseval's Identity.

6 Summary of Orthogonal Polynomial Systems

While Legendre polynomials are derived from the weight $w(x) = 1$ on $[-1, 1]$, other orthogonal systems arise from different weights using the same Gram-Schmidt logic.

Name	**Interval**	**Weight** $w(x)$	**Norm Square** $\|P_n\|^2$
Legendre $P_n(x)$	$[-1, 1]$	1	$\frac{2}{2n+1}$
Chebyshev $T_n(x)$	$[-1, 1]$	$(1 - x^2)^{-1/2}$	$\begin{cases} \pi & n = 0 \\ \pi/2 & n \neq 0 \end{cases}$
Laguerre $L_n(x)$	$[0, \infty)$	e^{-x}	1
Hermite $H_n(x)$	$(-\infty, \infty)$	e^{-x^2}	$\sqrt{\pi}2^n n!$