(4) General equation of wave

Consider a transverse wave in a string that starts at A and propagates from left to right along +ve x-direction as shown in fig-6. The particle on the right will start vibrating after a certain time as compared to the particle on the left. Since every particle of the medium performs simple harmonic motion, the equation of motion of any particle, say A, is given by

$$y = a \sin \omega t$$

where a is the amplitude of the vibrating particle, y is the displacement and angular velocity after time t.

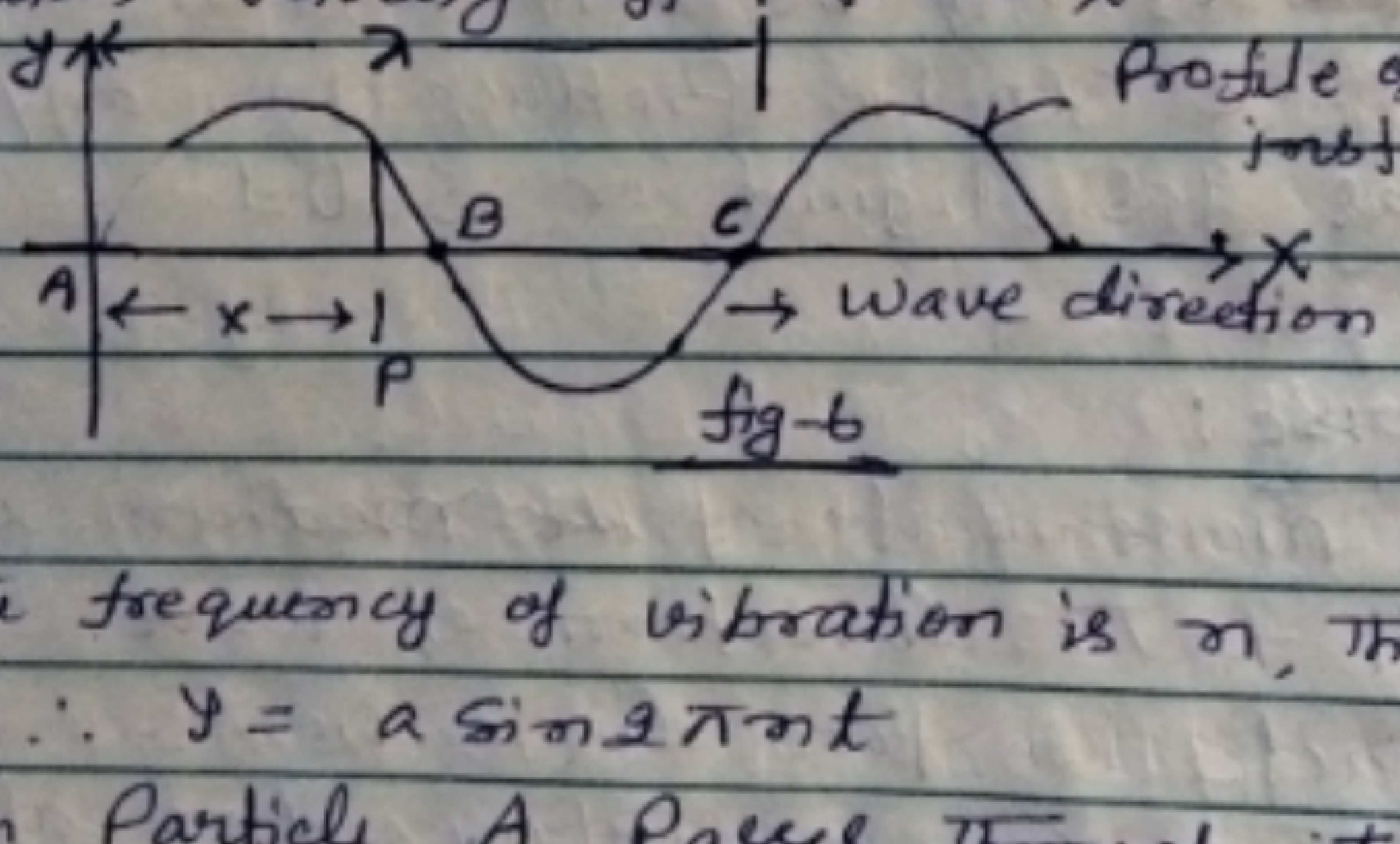

fig-6

If the frequency of vibration is n, then $\omega = 2\pi n$

$$\therefore \; y = a \sin 2\pi n t$$

When particle A passes through its mean position, then particles B, C, etc. also pass through their mean position in the same direction. So the particle A, B, C... etc are in the same phase. The distance between two consecutive particles in the same phase is called wavelength and on moving from A to B, the phase changes by 2π. Therefore, on moving from point A to point P at a distance x from A, the phase changes and is given by ϕ.

$$\phi = \frac{2\pi}{\lambda} x$$

Hence the displacement of P is given by:

$$y = a \sin(\omega t - \phi)$$

$$= a \sin\left(2\pi n t - \frac{2\pi}{\lambda} x\right) = a \sin\left(2\pi \frac{v}{\lambda} t - \frac{2\pi}{\lambda} x\right)$$

or $$y = a \sin \frac{2\pi}{\lambda}(vt - x) \quad \text{—— (1)}$$

The equation is quite general and gives the displacement of particles whose distance x from a fixed point A is known at any time. So ~~that~~ this is the equation for an increasingly simple harmonic wave. The number of wave-lengths in a unit distance is called wave-number. It is denoted by $\bar{v}$.

$$\therefore \bar{v} = \frac{1}{\lambda} \quad \text{—— (2)}$$

The quantity $2\pi/\lambda = K$ is called Propagation constant.

Other form: equ^n (1) can also be written as

$$y = a \sin\left(\frac{2\pi v}{\lambda} t - \frac{2\pi}{\lambda} x\right) = a \sin(\omega t - Kx) \quad \text{—— (3)}$$

$$\left[\because \frac{2\pi v}{\lambda} = 2\pi n = \omega\right]$$

In exponential form

$$y = ae^{i(\omega t - kt)} \quad \text{—— (4)}$$

The eqn $y = a \sin \frac{2\pi}{\lambda}(vt - x)$, $y = a \sin(\omega t - kx)$ and $y = ae^{i(\omega t - kx)}$ denote a wave moving to the right along +X axis.

If the wave moves to the left (along −X direction), the sign ϕ changes because the oscillations at X begin before that at X=0 and the equations are represented as:

$$\left.\begin{aligned} y &= a \sin \frac{2\pi}{\lambda}(vt + x), \\ y &= a \sin(\omega t + kx) \end{aligned}\right\} \quad \text{—— (5)}$$

and $y = ae^{i(\omega t + kx)}$ —— (6)

Differential equation of wave Motion

The displacement of a Particle in a medium is given by

$$y = a \sin \frac{2\pi}{\lambda}(vt - x) \quad \text{—— (1)}$$

where v is the velocity of the Progressive wave

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The velocity of the particle at any place at instant t is obtained by differentiating equn (1) with respect to time, we get

$$velocity = \frac{dy}{dt} = \frac{2\pi v}{\lambda} a \cos \frac{2\pi}{\lambda}(vt - x) \quad \text{--- (2)}$$

But $\frac{dy}{dx} = \frac{2\pi}{\lambda} a \cos \frac{2\pi}{\lambda}(vt - x)$ — (3)

Now dy/dx represents the strain on compression in the medium.

when dy/dx positive, rarefaction is occurs, and when -ve, compression occurs. From equn (2) and (3), we have

$$\frac{dy}{dt} = -v \frac{dy}{dx} \quad \text{--- (4)}$$

Particle velocity = wave velocity × slope of displacement curve or strain.

Thus, the velocity of the particle in a wave is not the same as the velocity of the wave. At any point x, at an instant t, the ~~wave~~ velocity of the particle (dy/dt) is -v times the slope of the wave displacement curve at the point. Differentiating the velocity dy/dt in equn (2) with respect to time, we get the acceleration as,

$$\frac{d^2y}{dt^2} = -\left(\frac{2\pi v}{\lambda}\right)^2 a \sin \frac{2\pi}{\lambda}(vt - x)$$

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$$\frac{d^2y}{dt^2} = -\left(\frac{2\pi v}{\lambda}\right)^2 a \sin \frac{2\pi}{\lambda}(vt - x)$$

$$= -\left(\frac{2\pi v}{\lambda}\right)^2 y = -\omega^2 y \quad \text{--- (5)}$$

As $\omega = 2\pi n = \frac{2\pi}{T} = \frac{2\pi v}{\lambda}$

where n is the frequency, T is the time period. From equ^n (5), it is also implied that the particle does S.H.M.

Differentiating equ^n (3), w.r.t x, we get

$$\frac{d^2y}{dx^2} = -\left(\frac{2\pi}{\lambda}\right)^2 a \sin \frac{2\pi}{\lambda}(vt - x)$$

or $$\frac{d^2y}{dx^2} = -\frac{\omega^2}{v^2} y \quad \text{--- (6)}$$

Comparing equ^n (5) and (6) we get

$$\frac{d^2y}{dt^2} = v^2 \frac{d^2y}{dx^2} \quad \text{--- (7)}$$

Thus, the acceleration of the particle is equal to v^2 times the curvature of the wave displacement curve. Equation (7) represents the differential equ^n of wave motion.

The general solution of equ^n (7) is

$y = f(vt - x) + g(vt + x)$

where f and g are any two functions. f denotes a plane wave moving in the +ve x-direction with velocity v, i.e. the forward wave and g represents a plane wave moving in the -ve x-direction. i.e. the back wave.