Transcendental Functions and Fundamental Theorems of Complex Analysis

1. Transcendental Functions

A transcendental function is a function that cannot be expressed as a finite combination of algebraic operations. In complex analysis, the most important transcendental functions are exponential, trigonometric, and hyperbolic functions.

1.1 Exponential Function

The exponential function for a complex variable $z = x + iy$ is defined as $e^z = e^x(\cos y + i \sin y)$. This definition is consistent with Euler's formula and extends the real exponential function to the complex plane.

Example: $e^{i\pi} + 1 = 0$, which is Euler's famous identity.

1.2 Trigonometric Functions

Complex trigonometric functions are defined using the exponential function as:
$\sin z = (e^{iz} - e^{-iz}) / 2i$,
$\cos z = (e^{iz} + e^{-iz}) / 2$.
These definitions reduce to the usual trigonometric functions for real values of z.

Example: $\sin(ix) = i \sinh x$ and $\cos(ix) = \cosh x$.

1.3 Hyperbolic Functions

Hyperbolic functions are defined as:
$\sinh z = (e^z - e^{-z}) / 2$,
$\cosh z = (e^z + e^{-z}) / 2$.
They play an important role in both real and complex analysis.

Example: $\cosh^2 z - \sinh^2 z = 1$.

2. Analytic Functions

A complex function f(z) is said to be analytic at a point if it is differentiable in a neighborhood of that point. If a function is analytic at every point of a domain, it is called an analytic function in that domain.

Example: $f(z) = z^2 + 3z + 2$ is analytic everywhere in the complex plane.

3. Cauchy-Riemann Equations

Let $f(z) = u(x, y) + iv(x, y)$ be a complex function. The necessary conditions for f(z) to be analytic are the Cauchy-Riemann equations:
$\partial u/\partial x = \partial v/\partial y$ and $\partial u/\partial y = -\partial v/\partial x$.

Example: For $f(z) = z^2$, $u = x^2 - y^2$ and $v = 2xy$ satisfy the Cauchy-Riemann equations.

4. Contour Integral

A contour integral is an integral of a complex function along a curve in the complex plane. If z(t) represents a contour C, then the contour integral of f(z) along C is defined as $\int_C f(z)\, dz$.

Example: $\int_C z\, dz$ along a straight line from 0 to 1+i equals $(1+i)^2 / 2$.

5. Cauchy’s Integral Theorem

Cauchy’s integral theorem states that if a function is analytic inside and on a simple closed contour C, then $\int_C f(z)\, dz = 0$.

Example: $\int_C z^2\, dz = 0$ for any closed contour C.

6. Cauchy Integral Formula

If f(z) is analytic inside and on a simple closed contour C and a is any point inside C, then:
$f(a) = (1 / 2\pi i) \int_C f(z) / (z - a)\, dz$.

Example: Using Cauchy’s integral formula, $\int_C dz/(z-a) = 2\pi i$.

7. Liouville’s Theorem

Liouville’s theorem states that every bounded entire function is constant. An entire function is a function that is analytic over the whole complex plane.

Example: If $f(z)$ is entire and $|f(z)| \leq 5$ for all z, then $f(z)$ must be a constant.