Differential Geometry of Space Curves

Introduction

Differential Geometry deals with the study of curves and surfaces using calculus. In this chapter, we focus on the geometry of space curves and their intrinsic properties such as curvature and torsion.

1. Tangent, Principal Normal and Binormal

Let $r(s)$ be a smooth curve parameterized by arc length s. The unit tangent vector is defined as $T = dr/ds$. The principal normal N is defined in the direction of dT/ds, and the binormal B is given by $B = T \times N$.

Example: For a circle of radius a, curvature is constant and the binormal vector is perpendicular to the plane of the circle.

2. Three Fundamental Planes

At any point on a space curve, three mutually perpendicular planes are defined:
• Osculating Plane – plane containing T and N
• Normal Plane – plane containing N and B
• Rectifying Plane – plane containing T and B

3. Curvature and Torsion of Space Curves

Curvature (κ) measures how fast a curve deviates from a straight line and is defined as $\kappa = |dT/ds|$. Torsion (τ) measures how fast the curve deviates from a plane and is given by $\tau = -dB/ds \cdot N$.

Example: A plane curve has zero torsion.

4. Serret–Frenet Formulae

The Serret–Frenet formulae describe the derivatives of T, N, and B with respect to arc length:
$dT/ds = \kappa N$
$dN/ds = -\kappa T + \tau B$
$dB/ds = -\tau N$

5. Fundamental Theorem of Space Curves

The fundamental theorem of space curves states that a curve is completely determined by its curvature and torsion, except for its position in space.

6. Helix

A helix is a space curve whose tangent makes a constant angle with a fixed direction. For a circular helix, curvature and torsion are constants.

Example: The curve $r(t) = (a \cos t, a \sin t, bt)$ represents a circular helix.

7. Spherical Indicatrices

The loci of the unit tangent, normal, and binormal vectors on the unit sphere are called spherical indicatrices.

8. Involute and Evolute

The involute of a curve is traced by the free end of a taut string as it is unwound from the curve. The evolute is the locus of centers of curvature of the curve.

Example: The evolute of a circle is its center.