Subject: Mathematics (Abstract Algebra)

Topic: Subgroups & Examples

1 Definition of a Subgroup

Let $(G, *)$ be a group. A non-empty subset H of G is called a **subgroup** of G if H itself forms a group under the same binary operation $*$ defined on G.

- **Notation:** We write $H \leq G$ to denote that H is a subgroup of G.
- If $H \leq G$ and $H \neq G$, then H is called a **proper subgroup** (denoted $H < G$).

2 Tests for Subgroups

To prove a subset is a subgroup, we do not need to check all four group axioms. We use the following criteria:

2.1 The Two-Step Subgroup Test

A non-empty subset $H \subseteq G$ is a subgroup if and only if:

1. **Closure:** For all $a, b \in H$, $a * b \in H$.
2. **Inverse:** For all $a \in H$, $a^{-1} \in H$.

2.2 The One-Step Subgroup Test

A non-empty subset $H \subseteq G$ is a subgroup if and only if:

$$\forall a, b \in H \implies a * b^{-1} \in H$$

(For additive groups: $a - b \in H$).

3 Types of Subgroups

1. **Trivial Subgroup:** The set $\{e\}$ consisting only of the identity element is always a subgroup of G.
2. **Improper Subgroup:** The group G itself is always a subgroup of G.
3. **Non-Trivial Subgroups:** Any subgroup containing elements other than just the identity.

4 Standard Examples of Subgroups

Example (**Number Systems**). Consider the groups under addition (+):

$$\mathbb{Z} \leq \mathbb{Q} \leq \mathbb{R} \leq \mathbb{C}$$

- The Integers $(\mathbb{Z}, +)$ are a subgroup of the Rationals $(\mathbb{Q}, +)$.
- The Rationals $(\mathbb{Q}, +)$ are a subgroup of the Reals $(\mathbb{R}, +)$.

Example (**Even Integers**). Let $2\mathbb{Z} = \{\dots, -4, -2, 0, 2, 4, \dots\}$ be the set of even integers.

- **Closure:** Sum of two even integers is even.
- **Inverse:** Negative of an even integer is even.
- Therefore, $(2\mathbb{Z}, +)$ is a subgroup of $(\mathbb{Z}, +)$.
- **Generalization:** For any integer n, the set $n\mathbb{Z}$ is a subgroup of $\mathbb{Z}$.

Example (**Matrix Groups**).

- Let $GL(2, \mathbb{R})$ be the group of all 2×2 invertible matrices.
- Let $SL(2, \mathbb{R})$ (Special Linear Group) be the set of matrices with determinant equal to 1.
- Since $\det(AB) = \det(A)\det(B) = 1 \cdot 1 = 1$, closure holds.
- Thus, $SL(2, \mathbb{R})$ is a subgroup of $GL(2, \mathbb{R})$.

5 Important Properties (Theorems)

Theorem 1 (**Intersection of Subgroups**). *If H and K are two subgroups of G, then their intersection $H \cap K$ is also a subgroup of G.*

Note. The **union** of two subgroups $H \cup K$ is **not** necessarily a subgroup (unless one is contained in the other).

Theorem 2 (**Center of a Group**). *The Center of a group $Z(G) = \{x \in G \mid xg = gx, \forall g \in G\}$ (elements that commute with everything) is always a subgroup of G.*