Groups (Important theorems and example) (Problem)

Theorem — If a and x are arbitrary elements of a group, then order of a is the same as the order of $x^{-1}ax$.

Proof. — Let a and x be arbitrary elements of a group G and e its identity element.

let $o(a) = n$, $o(x^{-1}ax) = m$

To prove that $o(a) = o(x^{-1}ax)$

i.e $m = n$ ——— (1)

$$(x^{-1}ax)^2 = (x^{-1}ax)(x^{-1}ax) = x^{-1}a(xx^{-1})ax = x^{-1}aeax$$
$$= x^{-1}aax = x^{-1}a^2x$$
$$\Rightarrow (x^{-1}ax)^2 = x^{-1}a^2x \quad \text{——— (2)}$$

i.e. $(x^{-1}ax)^3 \Rightarrow (x^{-1}ax)^2(x^{-1}ax) = (x^{-1}a^2x)(x^{-1}ax)$
$$\Rightarrow x^{-1}a^2(xx^{-1})ax = x^{-1}a^2(e)ax = x^{-1}a^2ax.$$
$$\Rightarrow (x^{-1}ax)^3 = x^{-1}a^3x \quad \text{——— (3)}$$

from eqn (2) and (3), We get :

$$(x^{-1}ax)^n = x^{-1}a^nx \quad \text{——— (4)}$$

Now,

$$o(a) = n \Rightarrow a^n = e$$
$$\Rightarrow (x^{-1}ax)^n = x^{-1}a^nx = x^{-1}ex = x^{-1}x = e$$
$$\Rightarrow (x^{-1}ax)^n = e \Rightarrow o(x^{-1}ax) \leq n$$
$$\Rightarrow m \leq n \quad \text{——— (5)}$$

Again,

$$o(x^{-1}ax) = m \Rightarrow (x^{-1}ax)^m = e$$

$\Rightarrow x^{-1}a^m x = e$ by eqn. (4)

$\Rightarrow x^{-1}a^m x = x^{-1}x \quad \because x^{-1}x = e.$

$\Rightarrow a^m x = x$, by left cancellation law

$\Rightarrow a^m x = ex$

$\Rightarrow a^m = e$, by right cancellation law.

$\Rightarrow o(a) \leq m \Rightarrow n \leq m$ ——— (6)

Combining eqn (5) and eqn. (6), we get

$n = m$

Proved.

Problem-(1) Prove that if $a^2 = a$, then $a = e$, a being an element of a group.

Solution — Let a be an element of a group G such that $a^2 = a$.

To prove that $a = e$.

$a^2 = a \Rightarrow a \cdot a = a \Rightarrow (a \cdot a)a^{-1} = aa^{-1}$

$\Rightarrow a(aa^{-1}) = e \quad \because aa^{-1} = e$

$\Rightarrow ae = e \Rightarrow a = e$

For $ae = a$.