Mathematical Methods

Topic: Bessel's Inequality & Approximation Theory

Level: M.Sc. Mathematics

Contents

1 Introduction: The Approximation Problem

In the theory of inner product spaces, a fundamental question is how to approximate an arbitrary vector f using a linear combination of a fixed set of orthonormal vectors $\{e_1, e_2, \ldots, e_n\}$.

Let V be an inner product space over the field $\mathbb{F}$ (where $\mathbb{F}$ is $\mathbb{R}$ or $\mathbb{C}$). Let $S = \{e_1, e_2, \ldots, e_n\}$ be a finite orthonormal set in V. This means:

$$\langle e_i, e_j \rangle = \delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} \tag{1}$$

Given an arbitrary vector $f \in V$, we seek to find the "best" approximation P in the subspace spanned by S:

$$P = \sum_{k=1}^{n} c_k e_k$$

The "best" approximation is defined as the one that minimizes the error norm $\|f - P\|$.

2 Least Squares Minimization

To minimize the error, we consider the square of the norm of the difference vector:

$$E = \|f - \sum_{k=1}^{n} c_k e_k\|^2$$

Theorem 2.1 (Best Approximation Theorem). *The error norm $\|f - \sum_{k=1}^{n} c_k e_k\|$ is minimized if and only if the coefficients c_k are chosen as the Fourier coefficients of f with respect to $\{e_k\}$:*

$$c_k = \langle f, e_k \rangle$$

Proof. Using the properties of the inner product norm $\|u\|^2 = \langle u, u \rangle$, we expand the error term:

$$\begin{aligned} E &= \langle f - \sum_{k=1}^{n} c_k e_k, f - \sum_{j=1}^{n} c_j e_j \rangle \\ &= \langle f, f \rangle - \langle f, \sum c_j e_j \rangle - \langle \sum c_k e_k, f \rangle + \langle \sum c_k e_k, \sum c_j e_j \rangle \end{aligned}$$

Using linearity in the first argument and conjugate linearity in the second:

$$E = \|f\|^2 - \sum_{j=1}^{n} \overline{c_j} \langle f, e_j \rangle - \sum_{k=1}^{n} c_k \langle e_k, f \rangle + \sum_{k=1}^{n} \sum_{j=1}^{n} c_k \overline{c_j} \langle e_k, e_j \rangle$$

Since the set is orthonormal, $\langle e_k, e_j \rangle = \delta_{kj}$, which collapses the double sum:

$$E = \|f\|^2 - \sum_{k=1}^{n} \overline{c_k} \langle f, e_k \rangle - \sum_{k=1}^{n} c_k \overline{\langle f, e_k \rangle} + \sum_{k=1}^{n} |c_k|^2$$

Let $\hat{c}_k = \langle f, e_k \rangle$ be the Fourier coefficient. We can rewrite the middle terms. Observe that $|c_k - \hat{c}_k|^2 = (c_k - \hat{c}_k)(\overline{c_k} - \overline{\hat{c}_k}) = |c_k|^2 - c_k\overline{\hat{c}_k} - \hat{c}_k\overline{c_k} + |\hat{c}_k|^2$. Rearranging our expression for E:

$$E = \|f\|^2 - \sum_{k=1}^{n} |\hat{c}_k|^2 + \sum_{k=1}^{n} |c_k - \hat{c}_k|^2 \tag{2}$$

Since norms and absolute values are non-negative, the quantity E is minimized when the last term is zero.

$$\sum_{k=1}^{n} |c_k - \hat{c}_k|^2 = 0 \iff c_k = \hat{c}_k$$

Thus, the optimal coefficients are $c_k = \langle f, e_k \rangle$. □

3 Bessel's Inequality

Bessel's Inequality is a direct consequence of the non-negativity of the error vector's norm. It provides an upper bound on the sum of the squared moduli of the Fourier coefficients.

3.1 Statement of the Inequality

Theorem 3.1 (Bessel's Inequality). *Let V be an inner product space and $\{e_1, e_2, \dots\}$ be an orthonormal sequence (finite or infinite) in V. For any $f \in V$, the sum of the squared absolute values of its Fourier coefficients is bounded by the squared norm of f:*

$$\sum_{k=1}^{\infty} |\langle f, e_k \rangle|^2 \leq \|f\|^2 \tag{3}$$

3.2 Proof (Finite Case)

Let n be a finite integer. We compute the squared norm of the "error vector" (the component of f orthogonal to the subspace):

$$u = f - \sum_{k=1}^{n} \langle f, e_k \rangle e_k$$

Since $\|u\|^2 \geq 0$, we have:

$$0 \leq \|f - \sum_{k=1}^{n} \langle f, e_k \rangle e_k\|^2$$

From our derivation in the Least Squares section (Eq. 2), with $c_k = \hat{c}_k$:

$$0 \leq \|f\|^2 - \sum_{k=1}^{n} |\langle f, e_k \rangle|^2$$

Rearranging gives the finite version of Bessel's Inequality:

$$\sum_{k=1}^{n} |\langle f, e_k \rangle|^2 \leq \|f\|^2 \tag{4}$$

3.3 Proof (Infinite Case)

If the orthonormal set is infinite $\{e_1, e_2, \dots\}$, consider the partial sums. Let $S_n = \sum_{k=1}^{n} |\langle f, e_k \rangle|^2$. From the finite case, we know that for any $n \in \mathbb{N}$:

$$S_n \leq \|f\|^2$$

The sequence of partial sums $\{S_n\}$ is:

1. **Monotonically Increasing:** Since each term $|\langle f, e_k \rangle|^2$ is non-negative.
2. **Bounded Above:** By $\|f\|^2$.

By the Monotone Convergence Theorem for sequences of real numbers, the limit as $n \to \infty$ exists and respects the bound:

$$\sum_{k=1}^{\infty} |\langle f, e_k \rangle|^2 \leq \|f\|^2$$

□

4 Geometric Interpretation

In Euclidean geometry ($\mathbb{R}^3$), if we project a vector $\mathbf{v}$ onto orthogonal axes x, y, z, the length of the vector is related to its components by the Pythagorean theorem: $|\mathbf{v}|^2 = v_x^2 + v_y^2 + v_z^2$.

If we project $\mathbf{v}$ onto a *subset* of axes (say, only x and y), the sum of squares of the components will be *less than or equal to* the total length squared:

$$v_x^2 + v_y^2 \leq |\mathbf{v}|^2$$

Bessel's Inequality is the infinite-dimensional generalization of this concept.

- The term $\sum |\langle f, e_k \rangle|^2$ represents the "energy" or "length" captured by the projection onto the orthonormal set.
- The term $\|f\|^2$ represents the total "energy" of the function.
- The inequality states that the energy of the projection cannot exceed the total energy of the original signal.

5 Convergence Implications

Bessel's Inequality has a profound consequence regarding the convergence of Fourier coefficients.

Corollary 5.1 (Riemann-Lebesgue Lemma). *If $\sum_{k=1}^{\infty} |\langle f, e_k \rangle|^2$ converges (which is guaranteed by Bessel's inequality), then the n-th term must approach zero as $n \to \infty$.*

$$\lim_{n \to \infty} \langle f, e_n \rangle = 0 \tag{5}$$

This means that for any square-integrable function, the Fourier coefficients must decay to zero for large n.

6 Parseval's Identity and Completeness

While Bessel's Inequality gives an upper bound ($\leq$), Parseval's Identity asserts equality ($=$).

Definition 6.1 (Complete Orthonormal Set). *An orthonormal set $\{e_k\}$ in a Hilbert space $\mathcal{H}$ is called **complete** (or a Basis) if the only vector orthogonal to all e_k is the zero vector.*

Theorem 6.1 (Parseval's Identity). *The equality holds in Bessel's Inequality:*

$$\sum_{k=1}^{\infty} |\langle f, e_k \rangle|^2 = \|f\|^2 \tag{6}$$

*if and only if the orthonormal set $\{e_k\}$ is **complete**.*

If the set is not complete, there exists some non-zero vector g orthogonal to all e_k. The projection of g onto the set would be 0, yielding $0 < \|g\|^2$, a strict inequality.

7 Example: Fourier Series

Consider the space $L^2[-\pi, \pi]$ with the orthonormal set:

$$\left\{ \frac{1}{\sqrt{2\pi}}, \frac{\cos(nx)}{\sqrt{\pi}}, \frac{\sin(nx)}{\sqrt{\pi}} \right\}_{n=1}^{\infty}$$

For a function $f(x) = x$, the Fourier coefficients a_n and b_n can be calculated. Bessel's inequality guarantees that:

$$\frac{a_0^2}{2} + \sum_{n=1}^{\infty} (a_n^2 + b_n^2) \leq \int_{-\pi}^{\pi} |x|^2 \, dx$$

Since the trigonometric system is complete, this actually becomes an equality (Parseval's), allowing us to sum infinite numerical series (e.g., finding the sum of $1/n^2$) by computing the integral on the right-hand side.