Important Problem of Groups

1. If G is a group of even order, then it has an element $a \neq e$ such that $a^2 = e$.

Solution: — Let G be a group of even order

To prove $\exists\ a \in G$ such that $a \neq e$, $a^2 = e$.

By group postulates, $a \in G \Rightarrow a^{-1} \in G$.

$O(G)$ is even $\Rightarrow \exists$ at least one element $a \in G$.

such that $a \neq e$, $a = a^{-1}$.

i.e. $\Rightarrow aa = aa^{-1} = e = a^2 = e$.

2. If $G = \{1, \omega, \omega^2\}$, then find order of every element of G. $1, \omega, \omega^2$ are cube roots of unity.

Solution: — Let $G = \{1, \omega, \omega^2\}$.

To determine the order of every element of G.

Here $e = 1 = \omega^3$.

$O(e) = 1$, for order of identity element of every group is one.

$\therefore\ O(\omega^3) = 1.$

To determine $O(\omega^2)$.

$\omega^3 = e$, lcm of 3 and 2 is 6

$O(\omega^2) = \frac{6}{2} = 3.$

To determine $O(\omega)$.

$\omega^3 = e$, lcm of 3 and 1 is 3.

$\therefore\ O(\omega) = \frac{3}{1} = 3.$

Finally, $O(1) = 1$, $O(\omega^3) = 3$, $O(\omega) = 3$.

3. If a and x are two elements of a group G such that $axa = b$, then find x.

Solution: — $axa = b \Rightarrow a^{-1}(axa) = a^{-1}b$

$\Rightarrow (a^{-1}a)(xa) = a^{-1}b$ (by associative law)

$\Rightarrow exa = a^{-1}b \Rightarrow xa = a^{-1}b$

$\Rightarrow (xa)a^{-1} = (a^{-1}b)a^{-1}$

$\Rightarrow x(aa^{-1}) = a^{-1}ba^{-1}.$

$\Rightarrow x = a^{-1}ba^{-1}$

For $aa^{-1} = e$.

$x = a^{-1}ba^{-1}$

Ans.