Discipline: **Physics**
Subject: **Electromagnetic Theory**
Unit 10:
Lesson/ Module: **Wave Guides and Cavities**

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Learning Objectives:

From this module students may get to know about the following:

1. *The boundary conditions to be satisfied in and around a perfect conductor.*
2. *The use of hollow metallic structures for "guided" propagation of electromagnetic waves.*
3. *The various modes of propagation in wave guides – The TE and TM modes and existence of cut-off frequencies.*
4. *Detailed study of rectangular wave guides for the two types of modes.*
5. *The cavity as a resonator – existence of resonant frequencies.*
6. *Detailed study of cylindrical cavity resonator and selection of the fundamental resonant frequency and tuning.*

Wave Guides and Cavities

10.1 Introduction

So far in the study of electromagnetic waves we have discussed waves of infinite extent, without any boundaries. We now wish to study waves confined in hollow spaces, viz. the interior of a hollow pipe. Study of such systems is of considerable practical importance. The only practical way of generating electromagnetic waves for wavelengths in the region of a few meters is through such metallic structures of size of the order of the wavelength. If the hollow pipe has end surfaces it is called a *cavity*, if not it is called a *waves guide*. We will first study fields in and outside of perfect conductors. Next we will study the propagation of waves in wave guides and in cavities. We will ignore complications due to finite conductivity which lead to attenuation of waves in the guides and broadening of the resonance in cavities.

10.2 Fields in and around a conductor

We will consider the case of a "perfect" conductor and an insulator interface. Since the conductor is perfect, $\vec{E}=0$ inside the conductor. Faraday's law then implies that $\frac{\partial \vec{B}}{\partial t}=0.$ If the magnetic field is zero to begin with, it remains zero throughout. Hence, inside the material of the waveguide, $\vec{B}=0$ as well. The null electric field inside the conductor is accomplished by instant movement of free charges on the surface of the conductor in response to any changes in the electric field. Such a change produces a surface charge density Σ so that the field inside the conductor remains zero:

$$\hat{n}.\vec{D}=\Sigma \tag{1}$$

where $\hat{n}$ is the unit normal drawn outward from the conducting surface. Similarly surface charges move in response to a changing magnetic field and produce surface currents $\vec{K}$ in such a way as to have a zero magnetic field inside the conductor:

$$\hat{n}\times\vec{H}=\vec{K} \tag{2}$$

The other two boundary conditions are on the normal component of $\vec{B}$ and tangential component of $\vec{E}$:

$$\begin{aligned}&\hat{n}.(\vec{B}_d-\vec{B}_c)=0;\\&\hat{n}\times(\vec{E}_d-\vec{E}_c)=0.\end{aligned} \tag{3}$$

The subscripts *c* and *d* refer to conductor and dielectric respectively.

10.3 Wave Guides and Cavities

As mentioned above, we will assume that the "guide" is a perfect conductor. For simplicity we assume that the size and shape of the general cylindrical waveguide is constant along the axis.

[Figure 9.23 from Griffiths. Change the shape of the pipe. Draw only z-axis. Change the shading.]

Let us consider sinusoidal time dependence $e^{-i\omega t}$ for the fields inside the hollow cylinder. In that case Maxwell equations take the form (make the replacement $\frac{\partial}{\partial t} \to -i\omega$ in Maxwell equations)

$$\begin{aligned} \vec{\nabla}.\vec{E} &= 0; \qquad & \vec{\nabla}\times\vec{E} &= i\omega\vec{B}; \\ \vec{\nabla}.\vec{B} &= 0; \qquad & \vec{\nabla}\times\vec{B} &= -i\mu\varepsilon\omega\vec{E} \end{aligned} \tag{4}$$

ε, μ are respectively the permittivity and permeability of the non-dissipative medium filling the cylinder. On taking the curl of either of the curl equations and using the other Maxwell equations above, it follows that electric and magnetic fields both satisfy the equation

$$(\nabla^2 + \mu\varepsilon\omega^2)\begin{Bmatrix}\vec{E}\\ \vec{B}\end{Bmatrix} = 0 \tag{5}$$

Let us take the axis of symmetry of the tube to be along the *z*-axis. Because of cylindrical symmetry it is convenient to separate out the *z*-dependence of the fields. We assume

$$\begin{Bmatrix}\vec{E}(\vec{x},t)\\ \vec{B}(\vec{x},t)\end{Bmatrix} = \begin{Bmatrix}\vec{E}(x,y)e^{i(\pm kz-\omega t)}\\ \vec{B}(x,y)e^{i(\pm kz-\omega t)}\end{Bmatrix} \tag{6}$$

These fields are of course complex; physical fields are the real parts. The wave vector *k* is taken to be real.

Let us separate the operator $\vec{\nabla}$ into transverse and *z*- components:

$$\vec{\nabla} = \vec{\nabla}_T + \hat{z}\frac{\partial}{\partial z}. \tag{7}$$

The Laplacian operator can then be written as

$$\nabla^2 = \nabla_T^2 + \frac{\partial^2}{\partial z^2}, \tag{8}$$

If we now substitute equation (6) into (5) and use equation (8), we obtain

$$[\nabla_T^2 + (\mu\varepsilon\omega^2 - k^2)]\begin{Bmatrix}\vec{E}\\ \vec{B}\end{Bmatrix} = 0 \tag{9}$$

We can take appropriate combinations of the two waves $e^{i(\pm kz-\omega t)}$ and obtain either traveling waves or standing waves as the situation demands.

Unlike in the case of waves in infinite free space, fields inside a hollow tube can have both transverse components as well as a component along the axis. It is convenient to separate the fields also into transverse and parallel components so that

$$\vec{E} = \vec{E}_T + \vec{E}_z; \qquad \vec{B} = \vec{B}_T + \vec{B}_z \tag{10}$$

$$\begin{aligned}\vec{E}_z &= E_z\hat{z}; \qquad & \vec{E}_T &= \vec{E} - \vec{E}_z = (\hat{z}\times\vec{E})\times\hat{z}\\ \vec{B}_z &= B_z\hat{z}; \qquad & \vec{B}_T &= \vec{B} - \vec{B}_z = (\hat{z}\times\vec{B})\times\hat{z}\end{aligned} \tag{11}$$

Using this decomposition of the fields and of $\vec{\nabla}$ operator [equation (7)] in the Maxwell's equations (4), we get

$$\frac{\partial\vec{E}_T}{\partial z} + i\omega\hat{z}\times\vec{B}_T = \vec{\nabla}_T E_z; \qquad \hat{z}.(\vec{\nabla}_T\times\vec{E}_T) = i\omega B_z; \tag{12}$$

$$\frac{\partial\vec{B}_T}{\partial z} - i\mu\varepsilon\omega\,\hat{z}\times\vec{E}_T = \vec{\nabla}_T B_z; \qquad \hat{z}.(\vec{\nabla}_T\times\vec{B}_T) = -i\mu\varepsilon\omega E_z; \tag{13}$$

$$\vec{\nabla}_T.\vec{E}_T = -\frac{\partial E_z}{\partial z}; \qquad \vec{\nabla}_T.\vec{B}_T = -\frac{\partial B_z}{\partial z} \tag{14}$$

With z-dependence given by equation (6) [we consider only e^{ikz} and not e^{-ikz}], we can replace $\frac{\partial}{\partial z}\to ik$. From the first part of equations (12) and (13) it follows that if z-components of the fields are known, the transverse components can be obtained from these equations:

$$\vec{E}_T = \frac{i}{\mu\varepsilon\omega^2 - k^2}[k\vec{\nabla}_T E_z - \omega\hat{z}\times\vec{\nabla}_T B_z] \tag{15}$$

$$\vec{B}_T = \frac{i}{\mu\varepsilon\omega^2 - k^2}[k\vec{\nabla}_T B_z + \mu\varepsilon\omega\hat{z}\times\vec{\nabla}_T E_z] \tag{16}$$

This is true only if z-components of electric and magnetic fields are not simultaneously zero; if that happens, it becomes the trivial null solution. In terms of the Cartesian components these equations can be written as

$$E_x = \frac{i}{(\mu\varepsilon\omega^2 - k^2)}(k\frac{\partial E_z}{\partial x} + \omega\frac{\partial B_z}{\partial y}) \tag{17}$$

$$E_y = \frac{i}{(\mu\varepsilon\omega^2 - k^2)}(k\frac{\partial E_z}{\partial y} - \omega\frac{\partial B_z}{\partial x}) \tag{18}$$

$$B_x = \frac{i}{(\mu\varepsilon\omega^2 - k^2)}(k\frac{\partial B_z}{\partial x} - \mu\varepsilon\omega\frac{\partial E_z}{\partial y}) \tag{19}$$

$$B_y = \frac{i}{(\mu\varepsilon\omega^2 - k^2)}(k\frac{\partial B_z}{\partial y} + \mu\varepsilon\omega\frac{\partial E_z}{\partial x}) \tag{20}$$

If $E_z = 0$, we have what are called *TE – Transverse Electric Waves*. On the other hand, if $B_z = 0$, we have *TM – Transverse Magnetic Waves*.

The third possibility is the degenerate case we have mentioned above in which both E_z and B_z are zero. This is the so called *TEM* mode. In this case the fields are transverse, i.e., they are two dimensional. On applying Gauss theorem and Faraday's law to this transverse electric field $\vec{E}_{TEM}$ we have

$$\vec{\nabla}.\vec{E}_{TEM} = 0; \qquad \vec{\nabla} \times \vec{E}_{TEM} = 0 \tag{21}$$

These are precisely the equations of electrostatics. This means that $\vec{E}_{TEM}$ is a solution of electrostatic problem in two dimensions. As a consequence it follows that the wave number for propagation along the axis is given by the free space value:

$$k = k_0 = \omega\sqrt{\mu\varepsilon}. \tag{22}$$

Secondly, for the *TEM* mode the magnetic field obtained from equation (13) is (remember $\frac{\partial}{\partial z} \to ik$)

$$ik\vec{B}_{TEM} - i\mu\varepsilon\omega\hat{z} \times \vec{E}_{TEM} = 0 \;\Rightarrow\; \vec{B}_{TEM} = \sqrt{\mu\varepsilon}\,\hat{z} \times \vec{E}_{TEM} \tag{23}$$

This is exactly the same as the relation for plane waves in infinite medium. Yet another consequence of the degenerate *TEM* mode is that waves corresponding to this mode cannot exist inside hollow wave guides. Since the vector $\vec{E}_{TEM}$ has zero divergence and zero curl it implies that it can be written as the gradient of a scalar potential that satisfies Laplace equation. But from the boundary condition $\vec{E}^{||} = 0$ it follows that the surface is an equipotential. Now solutions of Laplace equation have no local minima or maxima. Hence the potential must be a constant on the surface which implies that the electric field will necessarily be equal to zero. For the existence of the *TEM* mode it is necessary to have two or more cylindrical surfaces. Parallel wire transmission lines and coaxial cables are examples of structures of this kind. In fact in these cases the *TEM* mode is the dominant mode.

As we shall see soon, for the *TE* and *TM* modes there is a cutoff frequency below which waves cannot propagate; they are attenuated. For the *TEM* mode there is no cutoff frequency – wave number given by equation (22) is real for all frequencies.

10.3.1 The TM and TE modes

Let us now look at the various modes that are allowed in a hollow cylindrical wave guide or cavity. The longitudinal components of the fields satisfy wave equation (9) and certain boundary conditions. For time varying fields, as we have seen, both $(\vec{E},\vec{B})$ [equivalently $(\vec{D},\vec{H})$] vanish within a perfect conductor. Presence of surface charges and currents [equations (1) and (2)] allows a nonzero normal component of $\vec{D}$ and a tangential component of $\vec{H}$. But since the fields vanish identically inside the conductor, it follows from boundary conditions (3) that the tangential component of $\vec{E}$ and normal component of $\vec{B}$ must vanish at the boundary:

$$\hat{n}\times\vec{E}=0; \qquad \vec{n}.\vec{B}=0. \tag{24}$$

Since normal is perpendicular to the *z*-direction, the boundary condition on the electric field can be written as

$$E_z\big|_S=0. \tag{25}$$

Take the normal component of the first of equations (13). Along with the above boundary condition on electric field, it implies that the LHS of the equation is zero. This leads to the following boundary condition for the magnetic field:

$$\left.\frac{\partial B_z}{\partial n}\right|_S=0 \tag{26}$$

The two dimensional partial differential equation for the electric and magnetic fields, equation (9), together with the corresponding boundary condition (25) or (26) leads to an eigenvalue problem for the fields. For a given frequency ω only certain specified values of the wave number *k*, *the eigenvalues*, are allowed. This is the situation in waveguides where the frequency of the wave is specified and the wave number is determined from the eigenvalue problem. Equivalently, if we regard it as an eigenvalue problem in ω then for a given *k* only certain specified values of ω are allowed. This is the situation in resonant cavities in which only certain frequency modes can be excited.

Since the boundary conditions on E_z and B_z are different, the eigenvalues of the two will, in general, be different. The fields thus naturally divide themselves into two distinct categories:

Transverse Electric or *TE* waves

$$E_z \equiv 0; \qquad \left.\frac{\partial B_z}{\partial n}\right|_S = 0 \tag{27}$$

Transverse magnetic or *TM* waves

$$B_z \equiv 0; \qquad E_z\big|_S = 0. \tag{28}$$

Alternatively they are sometimes referred to as *magnetic* (or *M*) *waves* and *electric* (or *E*) *waves* respectively. These two, together with the *TEM* waves, wherever they can exist, form a complete set of solutions for the fields in a waveguide or a cavity.

10.4 Waveguides

Now let us look specifically at waveguides. It is more convenient (and perhaps conventional) to work with the magnetic field $\vec{H}$ than with $\vec{B}$. We take the *z*-dependence of the form e^{ikz} in equations (12) and (13). Then for the *TE* mode, $E_z = 0$ everywhere and from equation (12)

$$\vec{H}_T = \vec{B}_T / \mu = \frac{1}{Z}\hat{z} \times \vec{E}_T \tag{29}$$

For the *TM* mode $B_z = 0$ everywhere and we obtain the same result for $\vec{H}_T$ from equation (13) but for the definition of *Z*. The quantity *Z* is called the *wave impedance* and is given by

$$Z = \begin{cases} \dfrac{k}{\varepsilon\omega} = \dfrac{k}{k_0}\sqrt{\dfrac{\mu}{\varepsilon}} & (TM\ wave) \\ \dfrac{\mu\omega}{k} = \dfrac{k_0}{k}\sqrt{\dfrac{\mu}{\varepsilon}} & (TE\ wave) \end{cases} ; \qquad k_0 = \omega\sqrt{\mu\varepsilon} \tag{30}$$

For waves propagating in the opposite directions, *z*-dependence is given by e^{-ikz}. The results for this case can be obtained by the replacement $k \to -k$.

Once the *z*-components of the fields are determined from the boundary value problem, the transverse components are obtained from equations (15) and (16). If Ψ is the solution of the two dimensional wave equation

$$[\nabla_T^2 + K^2]\Psi = 0 \tag{31}$$

then for the two modes equations (15) and (16) yield

TE Waves

$$\vec{H}_T = \frac{ik}{K^2}\vec{\nabla}_T\Psi \tag{32}$$

TM waves

$$\vec{E}_T = \frac{ik}{K^2}\vec{\nabla}_T\Psi \tag{33}$$

where

$$K^2 = \mu\varepsilon\omega^2 - k^2 \tag{34}$$

The *TE* and *TM* waves are solutions of equation (31) corresponding respectively to the boundary conditions

$$\left.\frac{\partial\Psi}{\partial n}\right|_S = 0; \qquad \Psi\big|_S = 0. \tag{35}$$

Equation (31), along with one or the other of the boundary conditions (35), constitutes an eigenvalue problem. To satisfy the boundary conditions on opposite sides of the cylindrical surface, Ψ must be oscillatory which implies that K^2 must be positive. There is a set of eigenvalues, K_λ^2 and the corresponding eigenfunctions $\Psi_\lambda, \lambda = 1,2,3,\ldots$. These wavefunctions together form an *orthogonal set* and are the various *modes of the waveguide*. For a given frequency ω, there is a corresponding wave number k_λ given by

$$k_\lambda^2 = \mu\varepsilon\omega^2 - K_\lambda^2 \tag{36}$$

This can be written in the form

$$k_\lambda = \sqrt{\mu\varepsilon}\sqrt{\omega^2 - \omega_{\lambda c}^2} \tag{37}$$

where

$$\omega_{\lambda c} = K_\lambda / \sqrt{\mu\varepsilon} \tag{38}$$

is called the *cut-off frequency* corresponding to mode λ.

We now look at various features that follow from these equations:

- For $\omega > \omega_{\lambda c}$, the wave number k_λ is real: waves of mode λ. can propagate in the waveguide.
- For $\omega < \omega_{\lambda c}$, the wave number k_λ is imaginary and such waves cannot propagate in the waveguide and are called *cut-off* or *evanescent modes.*

- As frequency increases, it becomes higher than cut-off frequency for more and more values of λ.
- However, for any given frequency only a finite number of modes can propagate.
- The eigenfrequencies obviously depend on the dimensions of the waveguide, which are often so chosen that at the operating frequency only the lowest mode can propagate.

[See figure 8.4 from Jackson Ed.2. Replace the ordinate by $\frac{k_\lambda}{\sqrt{\mu\varepsilon}}$ instead of $\frac{ck_\lambda}{\sqrt{\mu\varepsilon}}$. Replace $\omega_1 \to \omega_{1c}; \omega_2 \to \omega_{2c}; \omega_3 \to \omega_{3c};$ and show only three instead of five curves. In the caption replace $\omega_\lambda \to \omega_{\lambda c}$.]

- The free space value of the wave number is $k = \sqrt{\mu\varepsilon}\omega$. From equation (37) it is clear that inside a waveguide the wave number is always less than its infinite space value. Consequently the phase velocity is always greater than the infinite space value:

$$v = \frac{\omega}{k_\lambda} = \frac{1}{\sqrt{\mu\varepsilon}} \frac{1}{\sqrt{1-(\omega_{\lambda c}/\omega)^2}} > \frac{1}{\sqrt{\mu\varepsilon}} \tag{39}$$

- This, however, is not inconsistent with theory of relativity, since the group velocity, the velocity at which energy or information is carried is always less than $\frac{1}{\sqrt{\mu\varepsilon}}$:

$$v_g = \frac{d\omega}{dk_\lambda} = \frac{1}{\sqrt{\mu\varepsilon}} \sqrt{1-\frac{\omega_{\lambda c}^2}{\omega^2}} < \frac{1}{\sqrt{\mu\varepsilon}} \tag{40}$$

- Exactly at the cut-off, phase velocity becomes infinite and the group velocity zero.

10.4.1 Modes in a rectangular waveguide

Let us now study in detail the specific case of a rectangular wave guide. We look at the TE waves; TM waves can be dealt with in exactly the same fashion. The length and height of the guide are *l* and *h* respectively. In Cartesian coordinates, wave equation (31) for $\Psi = H_z$ is

[Figure 8.5 from Jackson Ed. 2. Change symbols from a,b to l,h. Change design of the shaded area. Show *z*-axis also.]

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + K^2\right)\psi(x,y) = 0 \tag{41}$$

with boundary conditions $\frac{\partial \Psi}{\partial n} = 0$ at $x = 0, l$ and at $y = 0, h.$ On applying the usual method of separation of variables, $\Psi(x, y) = U(x)V(y)$ we get identical equations for function of *x* and *y*. The boundary conditions above take the form

$$\frac{dU}{dx} = 0, \quad x = 0, l; \qquad \frac{dV}{dy} = 0, \quad y = 0, h. \tag{42}$$

The solution of both equations is in terms of the trigonometric functions; on applying the boundary conditions we obtain the required solution

$$\Psi_{mn}(x, y) = H_0 \cos(\frac{m\pi x}{l}) \cos(\frac{n\pi y}{h}); \quad m, n = 0,1,2,3,..... \tag{43}$$

Instead of one single index representing the various modes we now have two indices which can independently take any integer value. Though one of the indices can be zero, both cannot be zero simultaneously since in that case the solution will be reduced to just a constant. Substituting back into the differential equation above we obtain

$$K_{mn}^2 = \pi^2 (\frac{m^2}{l^2} + \frac{n^2}{h^2}) \tag{44}$$

The cutoff frequencies for various modes are given by equation (38):

$$\omega_{mnc} = K_{mn} / \sqrt{\mu\varepsilon} = \frac{\pi}{\sqrt{\mu\varepsilon}} \sqrt{(\frac{m^2}{l^2} + \frac{n^2}{h^2})} \tag{45}$$

If *l* > *h*, the lowest frequency mode occurs for $m = 1, n = 0.$ This will be the dominant *TE* mode. The cutoff frequency for this mode is given by

$$\omega_{10c} = \frac{\pi}{\sqrt{\mu\varepsilon}} \frac{1}{l}. \tag{46}$$

The explicit fields for this mode, denoted by TE_{10} , are

$$\begin{aligned} H_z &= H_0 \cos(\frac{\pi x}{l}) e^{i(kz - \omega t)}; \qquad & H_x &= -\frac{ikl}{\pi} H_0 \sin(\frac{\pi x}{l}) e^{i(kz - \omega t)}; \\ E_z &= 0; & E_y &= \frac{i\mu\omega l}{\pi} H_0 \sin(\frac{\pi x}{l}) e^{i(kz - \omega t)} \end{aligned} \tag{47}$$

From equation (37)

$$k = \sqrt{\mu\varepsilon}\sqrt{\omega^2 - \omega_{10c}^2}. \qquad (48)$$

For the *TM* mode similar analysis shows that the field will be given in terms of sine rather than cosine functions; the lowest cutoff frequency will therefore correspond to $m = n = 1$. Hence the TE_{10} mode has the lowest cutoff frequency of all modes. This is the one which is used in most practical situations.

The ratio of lowest cutoff frequency to other cutoff frequencies depends upon the ratio of the two dimensions; *l/h*. As an illustration, for *l*=2*h*,

$$\omega_{mnc} = \frac{\pi}{\sqrt{\mu\varepsilon}}\sqrt{\left(\frac{m^2}{l^2} + \frac{4n^2}{l^2}\right)} \Rightarrow \frac{\omega_{mnc}}{\omega_{10c}} = \sqrt{m^2 + 4n^2} \qquad (49)$$

For this illustrative case, we represent this ratio in the form of a Table below:

$m \rightarrow$ $n \downarrow$	0	1	2	3	4	5
0	---	1	2	3	4	5
1	2	2.24	2.84	3.61	4.48	5.39
2	4	4.13	4.48	5.0	5.66	6.40
3	6	6.08	6.33	6.71	7.21	7.81
4	8	8.06	8.25	8.54	8.94	9.43

For this particular example, in the frequency range $\omega_{10c} < \omega < 2\omega_{10c}$, TE_{10} is the only mode that can propagate. In general this range will of course depend on the ratio *l/h*. At higher frequency more modes can propagate, the number crowding rapidly as the frequency increases.

10.5 Resonant Cavities

Whereas waveguides are meant to propagate electromagnetic waves, resonant cavities are structures that generate waves of specific frequencies. Most common resonator is a cylindrical wave guide closed at the ends. We assume that the end surfaces are plane and perpendicular to the axis of symmetry of the wave guide. As before, we assume further that the wall of the cavity is a perfect conductor while the hollow interior is filled with a dielectric with permittivity ε and permeability μ. The wave in the cavity is reflected at the ends, so that the *z*-dependence is taken to be one that is appropriate for the standing waves:

$$\Psi = A_1 \sin kz + A_2 \cos kz. \tag{50}$$

We take the length of the cavity to be *l*, and the two end surfaces to be the planes $z = 0, z = l$. Since the boundary conditions have to be satisfied at each surface, this turns into an eigenvalue problem with the solution

$$k = j\frac{\pi}{l}, \qquad j = 0,1,2,\ldots.. \tag{51}$$

For the *TE* fields, the boundary condition is vanishing of H_z at $z = 0, z = l$ and this leads to

$$H_z = \psi(x, y)\sin(j\frac{\pi z}{l}); \qquad j = 1,2,3\ldots. \tag{52}$$

For *TM* fields, from equation (15) the boundary condition is vanishing of $\frac{\partial E_z}{\partial z}$ at $z = 0, z = l$, and this leads to

$$E_z = \psi(x, y)\cos(j\frac{\pi z}{l}); \qquad j = 0,1,2,3\ldots. \tag{53}$$

The transverse fields are obtained by the application of equations (29), (32) and (33):

<u>*TE* Mode</u>

$$\begin{aligned} \vec{E}_T &= -i\frac{\omega\mu}{K^2}\sin(\frac{j\pi z}{l})\hat{z}\times\vec{\nabla}_T\psi; \\ \vec{H}_T &= \frac{j\pi}{lK^2}\cos(\frac{j\pi z}{l})\vec{\nabla}_T\psi \end{aligned} \tag{53}$$

<u>*TM* Mode</u>

$$\begin{aligned} \vec{E}_T &= -\frac{j\pi}{lK^2}\sin(\frac{j\pi z}{l})\vec{\nabla}_T\psi; \\ \vec{H}_T &= \frac{i\varepsilon\omega}{K^2}\cos(\frac{j\pi z}{l})\hat{z}\times\vec{\nabla}_T\psi \end{aligned} \tag{54}$$

The boundary conditions at the planar ends of the cavity are explicitly satisfied by the above construction.

The eigenvalue problem of equation (31) along with boundary conditions (35) remains to be solved. However, instead of equation (34) we now have

$$K^2 = \mu\varepsilon\omega^2 - (\frac{j\pi}{l})^2 \tag{55}$$

For each integer j, we obtain the eigenfrequency

$$\omega_{\lambda j} = \frac{1}{\sqrt{\mu\varepsilon}}[K_\lambda^2 + (\frac{j\pi}{l})^2]^{1/2} \tag{56}$$

The corresponding z-components of the fields are obtained from the solution of the differential equation (31) and the transverse components from the above equations. The resonant frequencies can be obtained from the Figure shown before. In that figure, take $k = \frac{j\pi}{l}$ and for every λ read the corresponding resonant frequency **[Refer to the figure on page 11 (Figure 8.4 from Jackson)]** $\omega_{\lambda j}$ along the x-axis. The resonant frequencies keep on crowding as j and λ increase. The dimensions of the cavity or so chosen that the desired frequency is well separated from other frequencies; the operation of the cavity is then stable against perturbations.

10.5.1 Cylindrical Resonant Cavity

Let us now look at the specific example of a right cylindrical cavity, which is the common type of cavity employed in practice. To allow for tuning, the length is usually kept variable by having a piston at one end of the cavity. Let the radius of the cavity be R and length be l. **[Figure 8.7 from Jackson edition 2. Name only the *z*-axis, *x* and *y* need not be named. Replace *d* by *l*.]** In cylindrical coordinates (ρ,φ), differential equation (31) becomes Bessel's equation. The relevant solution of the equation is in terms of Bessel functions of integer order, $J_m(x)$:

$$\psi(\rho,\phi) = J_m(K_{mn}\rho)e^{\pm im\varphi} \tag{57}$$

<u>*The TM mode*</u>

For the *TM* modes, the boundary condition is $\psi = E_z = 0$ at $\rho = R$. If x_{mn} is the n^{th} zero of $J_m(x)$ then the boundary condition implies

$$K_{mn}^{TM} = \frac{x_{mn}}{R}$$

$n \rightarrow$ $m \downarrow$	1	2	3	4
0	2.405	5.520	8.654	11.792
1	3.832	7.016	10.173	13.324
2	5.136	8.417	11.620	14.796
3	6.380	9.761	13.015	16.223

Some of the roots x_{mn} are tabulated above. Except for Bessel function of order zero, which has value unity at x=0, all other Bessel functions of integer order have a zero at x=0. That is a trivial case corresponding to null solution and has not been included in the Table above.

The resonant frequencies are given by

$$\omega_{mnj}^{TM} = \frac{1}{\sqrt{\mu\varepsilon}}[(\frac{x_{mn}}{R})^2 + (\frac{j\pi}{l})^2]^{1/2} \qquad (59)$$

These frequencies depend on both the radius and the length of the cavity. The lowest *TM* mode corresponds to $m = 0, n = 1, j = 0$. It is designated as TM_{010}. Its resonant frequency is

$$\omega_{010}^{TM} = \frac{1}{\sqrt{\mu\varepsilon}}\frac{x_{01}}{R} = \frac{1}{\sqrt{\mu\varepsilon}}\frac{2.405}{R} \qquad (60)$$

The lowest resonant frequency is independent of the length of the cavity and is therefore *not tunable by the piston*. The explicit expressions for the field are

$$\begin{aligned} E_z^{TM} &= E_0 J_0(\frac{2.405\rho}{R})e^{-i\omega t} \\ H_\varphi^{TM} &= -i\sqrt{\frac{\varepsilon}{\mu}}E_0 J_1(\frac{2.405\rho}{R})e^{-i\omega t} \end{aligned} \qquad (61)$$

<u>*The TE mode*</u>

The solution (57) applies to the *TE* mode as well. However the boundary condition now is $\left.\frac{\partial\psi}{\partial\rho}\right|_R = 0.$ Hence the eigenvalues are given in terms of the zeros of derivatives of Bessel functions. If x'_{mn} represents the n^{th} zero of the derivative of Bessel function, $J'_m(x)$, then from the boundary condition above, the eigenvalues are given by

$$K_{mn}^{TE} = \frac{x'_{mn}}{R}$$

$n \rightarrow$ $m \downarrow$	1	2	3	4
0	3.832	7.016	10.174	13.324
1	1.841	5.331	8.536	11.706
2	3.054	6.706	9.970	13.170
3	4.201	8.015	11.336	14.586

Some of the roots x'_{mn} are listed in the table above. For $J'_1(x)$ zero is also a root, which we have not listed since it is a trivial root corresponding to the null solution.

In this case the smallest root of a given order (even excluding the zero root) occurs for *m*=1, rather than *m*=0. The next roots in increasing order are for *m*=2, 0, 3, … From the roots follow the resonant frequencies, which in this case are

$$\omega_{mnj}^{TE} = \frac{1}{\sqrt{\mu\varepsilon}}[(\frac{x'_{mn}}{R})^2 + (\frac{j\pi}{l})^2]^{1/2}$$

The lowest *TE* mode is for $m = n = j = 1$ (in this case *j* begins from unity and not zero), and is denoted by TE_{111}. The resonant frequency is

$$\omega_{111}^{TE} = \frac{1}{\sqrt{\mu\varepsilon}}[(\frac{1.841}{R})^2 + (\frac{\pi}{l})^2]^{1/2}$$

The lowest frequency for the *TE* mode does depend upon the length of the cylinder and thus can be easily tuned to the desired value by making the length adjustable. It is therefore desirable to have ω_{111}^{TE} as the lowest resonance frequency. Comparing with the lowest *TM* mode frequency, we see that ω_{111}^{TE} becomes less than the lowest *TM* mode frequency, ω_{010}^{TM} for l>2.03R. Then ω_{111}^{TE} becomes the fundamental mode of the cavity and this mode is tunable by varying *l*. The field for this fundamental mode is given by

$$\psi = H_z = H_0 J_1(\frac{1.841\rho}{R})\cos\varphi \sin(\frac{\pi z}{l})e^{-i\omega t}$$

The other components can be obtained from equation (53).

Summary

1. *In this module we have first very briefly discussed the boundary conditions to be satisfied at the dielectric-perfect conductor interface.*
2. *We then explain the use of hollow metallic structures for the propagation of electromagnetic waves in a guided manner and resonant production of waves of suitable frequency.*
3. *Then we obtain the various modes of propagation in wave guides – The TE and TM modes. We explain why another mode, the TEM mode, is not possible in wave guides. We introduce the concept of cut-off frequencies below which propagation in a wave guide is not possible.*
4. *Next we make a detailed study of rectangular wave guides for the two types of modes and obtain the expressions for the cut-off frequencies for them.*
5. *Next we study a cavity resonator which is a wave guide like structure with the ends closed.*
6. *Finally we study specifically a cylindrical cavity and obtain expressions for the various resonant frequencies for both the TE and TM modes. The resonant frequencies are given in terms of the Bessel's functions and derivatives of Bessel's functions. We explain how it can be easily tuned to a desired frequency by making the length of the cylinder variable.*