# NOTES

## DEPARTMENT OF MATHEMATICS

### TOPIC:- MEAN ERROR MINIMIZATION

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ACADEMIC RESOURCES

# 1 Introduction: The Approximation Problem

Let $H$ be an Inner Product Space (e.g., a Hilbert Space like $L^2[a,b]$). Consider a function (or vector) $f \in H$. Let $S = \{\phi_1, \phi_2, \dots, \phi_n\}$ be a linearly independent set of vectors in $H$, and let $M = \text{span}(S)$ be the subspace spanned by $S$.

The **Approximation Problem** asks:

> Can we find a vector $g \in M$ such that the distance between $f$ and $g$ is minimized?

The distance is measured by the norm induced by the inner product:

$$\|f - g\| = \sqrt{\langle f - g, f - g \rangle}$$

We seek to find the coefficients $c_1, c_2, \dots, c_n$ such that the linear combination:

$$g = \sum_{k=1}^{n} c_k \phi_k$$

minimizes the **Mean Square Error**:

$$E(c_1, \dots, c_n) = \|f - \sum_{k=1}^{n} c_k \phi_k\|^2 \tag{1}$$

# 2 Derivation of the Minimum Error

We assume the set $\{\phi_k\}$ is **Orthonormal** (i.e., $\langle \phi_i, \phi_j \rangle = \delta_{ij}$). If the set is not orthonormal, we can first apply the Gram-Schmidt process.

## 2.1 Expansion of the Error Function

Let $g = \sum_{k=1}^{n} c_k \phi_k$. The squared error is:

$$\begin{aligned} E &= \|f - g\|^2 = \langle f - g, f - g \rangle \\ &= \langle f, f \rangle - \langle f, g \rangle - \langle g, f \rangle + \langle g, g \rangle \end{aligned}$$

Substituting the sum for $g$:

$$E = \|f\|^2 - \langle f, \sum c_k \phi_k \rangle - \langle \sum c_k \phi_k, f \rangle + \langle \sum c_j \phi_j, \sum c_k \phi_k \rangle$$

Using the linearity of the inner product:

$$E = \|f\|^2 - \sum_{k=1}^{n} \overline{c_k} \langle f, \phi_k \rangle - \sum_{k=1}^{n} c_k \langle \phi_k, f \rangle + \sum_{j=1}^{n} \sum_{k=1}^{n} c_j \overline{c_k} \langle \phi_j, \phi_k \rangle$$

Since the basis is orthonormal ($\langle \phi_j, \phi_k \rangle = \delta_{jk}$), the double sum collapses to a single sum:

$$E = \|f\|^2 - \sum_{k=1}^{n} \overline{c_k} \langle f, \phi_k \rangle - \sum_{k=1}^{n} c_k \overline{\langle f, \phi_k \rangle} + \sum_{k=1}^{n} |c_k|^2$$

### 2.2 Completing the Square

Let $\alpha_k = \langle f, \phi_k \rangle$ be the **Fourier Coefficients** of $f$ with respect to the orthonormal set. We observe that:

$$|c_k - \alpha_k|^2 = (c_k - \alpha_k)(\overline{c_k} - \overline{\alpha_k}) = |c_k|^2 - c_k\overline{\alpha_k} - \alpha_k\overline{c_k} + |\alpha_k|^2$$

Rearranging terms in our expression for $E$, we can write:

$$E = \|f\|^2 - \sum_{k=1}^{n} |\alpha_k|^2 + \sum_{k=1}^{n} |c_k - \alpha_k|^2 \tag{2}$$

### 2.3 Minimization Condition

The terms $\|f\|^2$ and $\sum |\alpha_k|^2$ are fixed constants determined by the function $f$ and the basis $\phi$. The only variable term is $\sum |c_k - \alpha_k|^2$. Since a sum of squares is always non-negative, $E$ is minimized when this term is zero.

$$c_k = \alpha_k = \langle f, \phi_k \rangle$$

**Theorem 2.1** (**Least Squares Approximation**). *The best approximation to $f$ in the mean square sense, using an orthonormal basis $\{\phi_k\}$, is obtained by choosing the Fourier coefficients $c_k = \langle f, \phi_k \rangle$. The minimum error is:*

$$E_{min} = \|f\|^2 - \sum_{k=1}^{n} |\langle f, \phi_k \rangle|^2$$

## 3 Geometric Interpretation: The Projection Theorem

The result above has a profound geometric meaning in Hilbert Space theory. Let $M$ be the subspace spanned by $\{\phi_1, \dots, \phi_n\}$. The vector $g^* = \sum \langle f, \phi_k \rangle \phi_k$ is the **Orthogonal Projection** of $f$ onto $M$.

**Theorem 3.1** (**Orthogonal Projection Theorem**). *Let $H$ be a Hilbert space and $M$ be a closed subspace. For every $f \in H$, there exists a unique element $g^* \in M$ such that:*

$$\|f - g^*\| = \inf_{g \in M} \|f - g\|$$

*Furthermore, the error vector $f - g^*$ is orthogonal to the subspace $M$:*

$$\langle f - g^*, y \rangle = 0 \quad \forall y \in M$$

*Proof.* Let $g^* = \sum_{k=1}^{n} \langle f, \phi_k \rangle \phi_k$. Consider any vector $\phi_j$ in the basis of $M$.

$$\begin{aligned}
\langle f - g^*, \phi_j \rangle &= \langle f, \phi_j \rangle - \langle \sum_{k=1}^{n} \langle f, \phi_k \rangle \phi_k, \phi_j \rangle \\
&= \langle f, \phi_j \rangle - \sum_{k=1}^{n} \langle f, \phi_k \rangle \delta_{kj} \\
&= \langle f, \phi_j \rangle - \langle f, \phi_j \rangle \\
&= 0
\end{aligned}$$

Since $f - g^*$ is orthogonal to every basis vector $\phi_j$, it is orthogonal to the entire subspace $M$. This orthogonality ensures that $g^*$ is the closest point in $M$ to $f$, analogous to dropping a perpendicular from a point to a plane in $\mathbb{R}^3$. □

# 4 Convergence and Completeness

As the number of basis functions $n$ increases, the subspace $M_n$ grows larger. We are interested in the limit as $n \to \infty$.

## 4.1 Bessel's Inequality Recalled

From the minimum error equation, since $E_{min} \geq 0$:

$$\|f\|^2 - \sum_{k=1}^{n} |\langle f, \phi_k \rangle|^2 \geq 0 \implies \sum_{k=1}^{n} |\langle f, \phi_k \rangle|^2 \leq \|f\|^2$$

This holds for any $n$, and thus for the infinite series.

## 4.2 Parseval's Identity

If the set $\{\phi_k\}_{k=1}^{\infty}$ is **complete** (a basis for the entire space $H$), then the minimum error approaches zero:

$$\lim_{n\to\infty} E_n = 0$$

This implies equality:

$$\sum_{k=1}^{\infty} |\langle f, \phi_k \rangle|^2 = \|f\|^2 \tag{3}$$

This is Parseval's Identity, confirming that the "energy" of the signal is preserved in the transform domain.

# 5 Example: Polynomial Approximation

Consider the space $L^2[-1, 1]$. We wish to approximate $f(x) = e^x$ using the first two Legendre Polynomials (orthonormalized). The orthonormal basis is:

$$\phi_0(x) = \frac{1}{\sqrt{2}}, \quad \phi_1(x) = \sqrt{\frac{3}{2}}x$$

**Step 1: Calculate Fourier Coefficients**

$$c_0 = \langle f, \phi_0 \rangle = \int_{-1}^{1} e^x \frac{1}{\sqrt{2}}\, dx = \frac{1}{\sqrt{2}}(e - e^{-1})$$

$$c_1 = \langle f, \phi_1 \rangle = \int_{-1}^{1} e^x \sqrt{\frac{3}{2}}x\, dx = \sqrt{\frac{3}{2}}\,[xe^x - e^x]_{-1}^{1} = \sqrt{\frac{3}{2}} \cdot 2e^{-1}$$

**Step 2: Construct the Approximation** The best linear approximation $P(x)$ is:

$$P(x) = c_0\phi_0(x) + c_1\phi_1(x)$$

This polynomial minimizes the integral of the squared difference $\int_{-1}^{1} |e^x - P(x)|^2 dx$.