Taylor’s Theorem and Maximum Modulus Principle

Detailed Explanation with Proofs and Examples

1. Taylor’s Theorem

Taylor’s theorem is a fundamental result in calculus and mathematical analysis which provides a powerful method of approximating functions by polynomials. It expresses a sufficiently smooth function as an infinite series of terms calculated from the values of its derivatives at a single point.

Statement of Taylor’s Theorem:

Let $f(x)$ be a function having derivatives of all orders in an interval containing a point a. Then for x in this interval, $f(x)$ can be expressed as

$f(x) = f(a) + (x-a)f'(a) + (x-a)^2/2!\, f''(a) + \ldots + (x-a)^n/n!\, f^{(n)}(a) + R_n$

where R_n is the remainder term which tends to zero as $n \to \infty$ under suitable conditions.

Proof of Taylor’s Theorem:

Consider a function $f(x)$ which is $(n+1)$ times differentiable in an interval containing a. Using repeated application of the Mean Value Theorem, we expand the function about $x = a$. By constructing a polynomial $P_n(x)$ that matches the function and its derivatives at $x = a$, and analyzing the error term, we obtain the Taylor expansion. The remainder term R_n is given by

$R_n = (x-a)^{(n+1)}/(n+1)!\, f^{(n+1)}(\xi)$, where ξ lies between a and x.

As n increases, if $f^{(n+1)}(\xi)$ remains bounded, the remainder term approaches zero, and hence the Taylor series converges to $f(x)$.

Examples:

1) Expansion of e^x about $x = 0$ (Maclaurin series): $e^x = 1 + x + x^2/2! + x^3/3! + \ldots$

2) Expansion of sin x about $x = 0$: $\sin x = x - x^3/3! + x^5/5! - \ldots$

2. Maximum Modulus Principle

The Maximum Modulus Principle is a central result in complex analysis. It describes the behavior of analytic functions and places strong restrictions on where a function can attain its maximum absolute value.

Statement of Maximum Modulus Principle:

If f(z) is a non-constant analytic function in a bounded domain D and continuous on its closure, then |f(z)| attains its maximum value on the boundary of D, and not at any interior point.

Proof of Maximum Modulus Principle:

Assume that |f(z)| attains a maximum at an interior point z■ of the domain D. Since f(z) is analytic and non-constant, it admits a power series expansion about z■. If the maximum occurs at z■, then all coefficients of the power series beyond the constant term must be zero, implying that f(z) is constant. This contradicts the assumption that f(z) is non-constant. Hence, the maximum modulus cannot occur in the interior of D.

Examples:

1) Let $f(z) = z^2$ in the unit disk $|z| < 1$. Then $|f(z)| = |z|^2 < 1$ inside the disk, and the maximum value occurs on $|z| = 1$.

2) For f(z) = e^z in the region |z| ≤ 1, the maximum of |e^z| occurs on the boundary of the region.

Thus, the Maximum Modulus Principle highlights the rigidity of analytic functions and plays a crucial role in many important results of complex analysis.