

Discipline: Physics
Subject: Electromagnetic Theory
Unit 14:
Lesson/ Module: Theory of Relativity - I

Author (CW): Prof. V. K. Gupta
Department/ University: Department of Physics and Astrophysics,
University of Delhi, New Delhi-110007



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Learning Objectives:

From this module students may get to know about the following:

- 1. Many crises faced by physics at the end of the nineteenth century. The resolution of the problem of light leading to the formulation of the special theory of relativity.*
- 2. The Michelson – Morley experiment and the importance of the null result of the experiment.*
- 3. Postulates of the theory of relativity.*
- 4. Lorentz transformations of the space-time coordinates as viewed in different moving frames of reference.*
- 5. The revolutionary and counter-intuitive implications of the consequences of Lorentz transformations of the space-time coordinates – length contraction and time dilatation.*
- 6. The relativistic Doppler effect.*



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I4. Theory of Relativity - I

14.1 Introduction

The world of physics faced two very special problems in the last quarter of the nineteenth century: one was theoretical and concerned radiation by a black body, and the other was experimental and concerned certain velocity of light measurements by Michelson and Morley in 1887. The subsequent solving of these two problems led to a double revolution in physics which changed the face of physics for ever. The first problem, the problem of black body radiation, led eventually to the birth of *Quantum Theory*. What concerns us here is the second problem, highlighted by the null result of a certain experiment due to Michelson and Morley, which was solved by Einstein and led to the *Special Theory of Relativity* and eventually to the *General Theory of Relativity*.

After Maxwell wrote down the equations of electromagnetism, which admitted of wave-like solutions with velocity equalling the velocity of light, the subjects of optics and electromagnetism were truly synthesized. Other waves, such as sound waves and water waves need a medium for them to be transmitted – they cannot travel through vacuum. It was natural to believe that a medium was necessary for transmission of light, that is, electromagnetic waves, as well. This medium was called *ether, aether or luminiferous ether* which filled up all space and “vibrated” as light waves passed through it. To get the correct value of velocity of light, this medium ether needed to have some peculiar properties – negligible density, high elasticity and negligible interaction with matter; as if it was created for the sole purpose of providing a medium for propagation of light waves.

However peculiar its properties, if it exists, we must be able to perceive the motion of material bodies with respect to it. *All attempts to find this relative motion failed completely.* Michelson Morley experiment was one such experiment and its negative result proved the last nail in the coffin of theory of ether.

It was known that the laws of mechanics were the same in various coordinate systems moving uniformly with respect to each other. The magnitude of various physical quantities is of course different in different frames. They are connected by what are called *Galilean Transformations*. Consider two frames of reference K and K' having a common origin at time $t = 0$ and K' moving with velocity \vec{v} with respect to K . The relations between the mechanical quantities in the two frames are given by

$$\vec{x}' = \vec{x} - \vec{v}t; \quad t' = t.$$

If the velocity of a particle as measured in K and K' is \vec{u} and \vec{u}' respectively, then from the above given Galilean transformations it follows that

$$\vec{u}' = \vec{u} - \vec{v},$$

the well-known relation between the velocities in two moving frames.

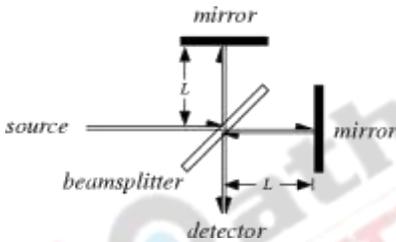
14.2 The Michelson-Morley Experiment

Accepting the existence of ether, the same formula should apply to light passing through ether as well. In the experiment conducted by Michelson and Morley, light from a source falls on a half-silvered mirror (beam splitter) set at 45° ; is split into two halves; falls on two mirrors at normal incidence, retrace their path to the half-silvered mirror and then onto a Michelson interferometer. Let the light ray take a time t to move from the splitter to the mirror above and back. Since its velocity is v , the mirror travels a distance vt . Hence

$$L^2 + (vt/2)^2 = (ct/2)^2$$

or

$$t = \frac{2L}{\sqrt{c^2 - v^2}}.$$



The second light ray travels a distance L with velocity $c-v$ and then travels back a distance L with velocity $c+v$. Hence the total time taken by this ray is

$$t' = \frac{L}{c-v} + \frac{L}{c+v} = \frac{2Lc}{c^2 - v^2} \neq t$$

If the two mirrors are rotated through 90° , t and t' are interchanged and this should lead to a shift in the interference pattern. The experiment however, showed no shift in the interference fringe pattern implying that the two times are exactly the same.

The problem can be looked at in a slightly different way. Whereas laws of mechanics were known to be invariant under Galilean transformations, wave propagation, in particular, electromagnetic wave propagation is not. Thus we either have to accept that Galilean transformations apply only to mechanics; for electromagnetic phenomena there is a preferred frame, viz., the rest frame of ether. However, as we have seen above, all attempts to find motion relative to the ether failed altogether and there was a crisis at hand. Many attempts were made to find an explanation of the negative results of the Michelson-Morley experiment within the ether hypothesis but none of them was really satisfactory.

14.3 Postulates of Theory of Relativity

To Einstein symmetry and simplicity of a theory mattered more than anything else. He rejected the rather clumsy attempts at explaining the negative results of all experiments on the relative

velocity of matter and ether. He in fact rejected the very hypothesis of ether lock, stock and barrel and boldly put forward his new postulates of relativity with far reaching and revolutionary consequences, which along with the revolution caused by quantum theory were to turn physics upside down on its head.

The two postulates put forward by Einstein are:

- Principle of relativity: The laws of nature are the same in all inertial frames of reference.
- Constancy of velocity of light: The speed of light in vacuum is independent of the motion of its source. That is, the velocity of light is the same in all inertial frames of reference.

Here we have introduced the term *inertial frames of reference*. These are the frames of reference in which Newton's laws are valid. If one has two observers, one moving with respect to the other with uniform velocity, then the laws of mechanics, optics or electrodynamics established by doing experiments by both observers will be same.

If one accepts this revolutionary idea of Einstein, the laws of mechanics need to be modified. At the time Einstein made this proposal in 1905, there was no evidence whatsoever of the failure of Galilean relativity for mechanics. If Newtonian mechanics indeed needed modifications, how come no evidence of there being anything wrong with it came out? The reason for this, as we shall see later, is that in mechanical phenomena the modification introduced by Einstein's relativity are of order v^2/c^2 , and thus become important at extremely high speeds, which are only now routinely attained in accelerators and other high-energy experiments.

14.4 Minkowski Space and Lorentz Transformations

The hypothesis of the constancy of velocity of light, independent of the motion of the source, gives rise to relations between space and time coordinates in different inertial frames of reference known as *Lorentz Transformations*. As we shall see, these relations inextricably mix together the space and time coordinates, so that it is much more convenient to think of the two together. Minkowski, in 1908, used the space-time approach to relativity by introducing time as the fourth dimension in a space time continuum. The four-dimensional space is called *Minkowski space or space-time*. Any point in this space will have four coordinates: x, y, z, t . Any point in this Minkowski space is called an *event*. The event may or may not be the actual occurrence of a physical phenomenon at a given point in space at a given time. Minkowski space is thus the set of all possible events.

We choose a frame of reference K in which coordinates are denoted by (x, y, z, t) and a frame K' in which the coordinates are denoted by (x', y', z', t') . For simplicity we choose the origin of space and time in the two frames to coincide at time $t=t'=0$. [See Figure] The frame K' is moving in the positive x -direction with speed v with respect to the frame K . Consider a ray of light moving parallel to the x -axis. Then such a ray obeys the equation

$$x = \pm ct$$

depending on the direction of propagation of light. In both cases

$$x^2 = c^2 t^2 \quad \text{or} \quad x^2 - c^2 t^2 = 0.$$

Similarly, for a spherical light ray

$$x^2 - y^2 - z^2 - c^2 t^2 = 0 \quad (1)$$

For the same event as viewed in the K' frame, we have

$$x'^2 - y'^2 - z'^2 - c^2 t'^2 = 0$$

since the velocity of light is the same in the two frames. It follows from the above that

$$x^2 - y^2 - z^2 - c^2 t^2 = \lambda(x'^2 - y'^2 - z'^2 - c^2 t'^2).$$

Since the relation must hold for the inverse transformation as well, $\lambda^2=1$, or $\lambda=\pm 1$. The fact that it must of course hold for identity transformation implies $\lambda=1$, and

$$x^2 - y^2 - z^2 - c^2 t^2 = x'^2 - y'^2 - z'^2 - c^2 t'^2 \quad (2)$$

We shall now *assume* that the Lorentz transformations are a set of linear transformations. We have thus to find a linear transformation under which the quantity $x^2 - y^2 - z^2 - c^2 t^2$ remains invariant.

Assume for the time being that there is only one space dimension x . Then the requirement reduces to

$$x^2 - c^2 t^2 = x'^2 - c^2 t'^2. \quad (3)$$

Now this is something like rotation in a plane except that instead of $x^2 + c^2 t^2$, the quantity that remains invariant is $x^2 - c^2 t^2$. Hence we try the transformation

$$x' = \cosh(\mathcal{G})x + \sinh(\mathcal{G})ct \quad (4)$$

$$ct' = \sinh(\mathcal{G})x + \cosh(\mathcal{G})ct \quad (5)$$

Look at the origin of the K' frame of reference. In the K' frame of reference, its coordinates are $x'=0$. Looked at from the K frame of reference its coordinates are given by $x=vt$. Substituting these values in (4), we have

$$0 = \cosh(\mathcal{G})vt + \sinh(\mathcal{G})ct \Rightarrow \tanh(\mathcal{G}) = -v/c.$$

From this we obtain, using the familiar relations for the hyperbolic functions

$$\cosh(\mathcal{G}) = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$$\sinh(\theta) = -\frac{v}{c} \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}.$$

Using these values in (4) and (5) we obtain

$$x' = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} (x - vt), \quad (6)$$

$$t' = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \left(-\frac{v}{c^2}x + t\right), \quad (7)$$

The following notation has become standard in the theory of relativity

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}, \quad \beta = v/c \quad (8)$$

Writing in terms of γ and β , these equations can be written as

$$x' = \gamma(x - vt), \quad t' = \gamma\left(-\frac{v}{c^2}x + t\right). \quad (9)$$

or

$$x' = \gamma(x - \beta ct), \quad ct' = \gamma(-\beta x + ct). \quad (11)$$

The *inverse Lorentz transformations* take us from the frame K' to K . These can be obtained by simply replacing $(x, t) \Leftrightarrow (x', t')$ and $v \rightarrow -v$:

$$x = \gamma(x' + vt'), \quad t = \gamma\left(\frac{v}{c^2}x' + t'\right)$$

or

$$x = \gamma(x' + \beta ct'), \quad ct = \gamma(\beta x' + ct'). \quad (12)$$

Returning now to our four dimensional space this special Lorentz transformation in which the relative motion of the two frames of reference is along the x -axis can be written as

$$x' = \gamma(x - vt), \quad t' = \gamma\left(-\frac{v}{c^2}x + t\right), \quad y' = y, \quad z' = z. \quad (13)$$

The inverse transformation is, similarly,

$$x = \gamma(x' + vt'), \quad t = \gamma\left(\frac{v}{c^2}x' + t'\right), \quad y = y', \quad z = z'. \quad (14)$$

The time coordinate and the space coordinate along the direction of motion are transformed, whereas the transverse space coordinates remain unchanged. In case the relative velocity \vec{v} of the two frames is along a general direction, the resultant Lorentz transformation is

$$\begin{aligned} t' &= \gamma\left(t - \frac{\vec{v} \cdot \vec{x}}{c^2}\right) = \gamma\left(t - \frac{\vec{\beta} \cdot \vec{x}}{c}\right) \\ \vec{x}' &= \vec{x} + \frac{\gamma - 1}{v^2}(\vec{v} \cdot \vec{x})\vec{v} - \gamma\vec{v}t = \vec{x} + \frac{\gamma - 1}{\beta^2}(\vec{\beta} \cdot \vec{x})\vec{\beta} - \gamma\vec{\beta}t/c \end{aligned} \quad (15)$$

Equivalently

$$\begin{aligned} t' &= \gamma\left(t - \frac{\vec{\beta} \cdot \vec{x}}{c}\right) \\ \vec{x}'_{\parallel} &= \gamma(\vec{x}_{\parallel} - \beta ct) \\ \vec{x}'_{\perp} &= \vec{x}_{\perp} \end{aligned} \quad (16)$$

14.5 Four vectors

In order to unify space and time into a four-dimensional setting, we will denote the space-time coordinates (t, x, y, z) by (x^0, x^1, x^2, x^3) , where

$$x^0 = ct, \quad x^1 = x, \quad x^2 = y, \quad x^3 = z. \quad (17)$$

Notice that x^0 is not t but ct . As a result all four quantities have the same dimensions. In terms of these coordinates Lorentz transformations can be written as

$$x^0' = \gamma(-\beta x^1 + x^0), \quad x^1' = \gamma(x^1 - \beta x^0), \quad x^2' = x^2, \quad x^3' = x^3 \quad (18)$$

The inverse transformations become

$$x^0 = \gamma(\beta x^1' + x^0'), \quad x^1 = \gamma(x^1' + \beta x^0'), \quad x^2 = x^2', \quad x^3 = x^3' \quad (19)$$

A Greek symbol is used when we refer to all four quantities together, and Latin symbol when we want to refer to the three space coordinates. Thus $x_i = (x_1, x_2, x_3)$ and $x^\mu = (x_0, \vec{x})$. We have used μ as a superscript. The distinction between four-vectors with subscripts and superscripts can be ignored for the time being, but will soon be made clear.

In three dimensions we call \vec{x} a vector and (x_1, x_2, x_3) the components of vector \vec{x} . Any set of three physical quantities that transform under rotations like the components of \vec{x} is called a

vector. We then extend this concept to tensors of second and higher ranks. There indeed are quantities that transform under rotations like a scalar (mass, energy), vectors (velocity, momentum, force, electric and magnetic fields) and tensors of higher ranks (pressure, stress, strain).

Similarly one expects many physical quantities that transform under Lorentz transformations like the space and time coordinates of a point. By analogy, we can speak of *four-vectors*. The coordinates x^0, x^1, x^2, x^3 , together form a four vector. By analogy, any set of four quantities that transform under Lorentz transformations like the components x^μ of space-time coordinates are four-vectors.

14.6 Length contraction and Time Dilatation

Before developing the mathematical structure of space-time in relativity, we first look at some very interesting, far-reaching and revolutionary consequences that follow from Lorentz transformations.

For simplicity for the moment we shall continue to work with only one space dimension so that our Lorentz transformations from K to K' take the form of equation (9):

$$x' = \gamma(x - vt), \quad t' = \gamma\left(-\frac{v}{c^2}x + t\right). \quad (20)$$

And the inverse transformation

$$x = \gamma(x' + vt'), \quad t = \gamma\left(\frac{v}{c^2}x' + t'\right) \quad (21)$$

Consider two events (x_1, t_1) and (x_2, t_2) in the frame K . If they are *simultaneous* in K , then $t_1 = t_2$. From (15), we have, in the frame K'

$$t_1' - t_2' = \gamma\left(-\frac{v}{c^2}\right)(x_1 - x_2) \neq 0$$

since $(x_1 - x_2) \neq 0$. Hence two events that are simultaneous in one frame are not in general simultaneous in some other frame of reference. *Simultaneity thus becomes a relative concept.* This is totally different from Newtonian mechanics where time flow at the same rate in all frames and events that are simultaneous in one frame are simultaneous in all frames.

Again consider the frames K and K' and a rod of length L' lying along x' -axis in K' . Let the coordinates of the two end points of the rod be $(x_1, t_1), (x_2, t_2)$ in K and $(x_1', t_1'), (x_2', t_2')$ in K' , so that $L = (x_2 - x_1)$ and $L' = (x_2' - x_1')$. We *define* the length of a moving rod to be the distance obtained by measuring the positions of the two ends of the rod *simultaneously*, as if a snapshot of the rod is taken. Thus in this measurement $t_1 = t_2$, and from (15) again we have

$$L' = (x_2' - x_1') = \gamma(x_2 - x_1) = \gamma L \quad \text{or} \quad L = L' / \gamma.$$

Thus the rod appears to be *contracted* in the moving frame by a factor γ . This result becomes important at extremely high speeds only; the kind obtained in high energy accelerators.

Using the inverse transformation (16) it is easy to see that a rod that is stationary in K will appear contracted in K' by the same factor γ .

Now consider a clock at rest in K' and an observer in K record two times: (t'_1, t'_2) . An observer in K reads the values (t_1, t_2) for the two times. So the space-time coordinates of the clock are respectively $(x_1, t_1), (x_2, t_2)$ in K and $(x'_1, t'_1), (x'_2, t'_2)$ in K' . $x'_1 = x'_2$, since the clock is at rest in K' . From the inverse transformations (16) we easily obtain

$$(t_2 - t_1) = \gamma(t'_2 - t'_1), \quad (22)$$

Similarly infinitesimal time intervals Δt and $\Delta t'$ in the two frames are connected by

$$\Delta t = \gamma \Delta t'.$$

Δt is the interval between two events in the moving frame and $\Delta t'$ is the interval between same two events in the frame at rest. Δt is a factor γ larger than $\Delta t'$ - an observer in K records longer time intervals; time dilates in the moving frame.

As in the case of length contraction, the result is symmetric between the two frames, i.e., a moving clock in K will appear slow in the frame K' .

14.7 The Relativistic Doppler Shift

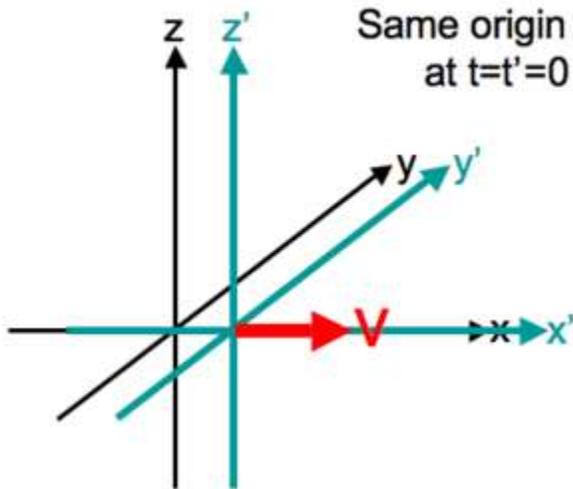
The relativistic Doppler effect is the *change in frequency* (and wavelength) of light, caused by the relative motion of the source and the observer (as in the classical Doppler effect), when taking in to account effects described by the special theory of relativity.

The relativistic Doppler effect is different from the non-relativistic Doppler effect as the equations include the time dilation effect of special relativity and do not involve the medium of propagation as a reference point.

Let there be two inertial frames of reference, K and K' , constructed so that the axes of K and K' coincide at $t = t' = 0$, where t is the time as measured in K and t' is the time as measured in K' . Let K' be in motion relative to K with constant velocity v ; without loss of generality, we will take this motion to be directed only along the x-axis. Thus, the Lorentz transformation equations take the form:

$$\begin{aligned} x &= \gamma(x' + \beta ct'), \\ ct &= \gamma(\beta x' + ct'), \\ y &= y', \\ z &= z' \end{aligned}$$

$$\frac{dx}{dt} = \frac{\frac{dx'}{dt'} + v}{1 + \frac{v}{c^2} \frac{dx'}{dt'}}$$



The derivation begins with what the observer in K' trivially sees. We imagine a signal source is positioned stationary at the origin, O' , of the K' system. We will take this signal source to produce its first pulse at time $t'_1 = 0$ (this is *event 1*) and its second pulse at time $t'_2 = 1/f'$ (this is *event 2*), where f' is the frequency of the signal source as the observer in K' sees it. We then simply use the Lorentz transformation equations to see when and where the observer in K sees these two events as occurring:

	<i>Observer in K'</i>	<i>Observer in K</i>
Event 1	$x'_1 = 0$	$x_1 = 0$
	$t'_1 = 0$	$t_1 = 0$

	<i>Observer in K'</i>	<i>Observer in K</i>
Event 2	$x'_2 = 0$	$x_2 = \gamma \frac{v}{f'}$
	$t'_2 = \frac{1}{f'}$	$t_2 = \gamma \frac{1}{f'}$

The period between the pulses as measured by the K observer is *not*, however, $t_2 - t_1$ because *event 2* occurs at a different point in space to *event 1* as observed by the K observer (that is, $x_2 \neq x_1$) — we must factor in the time taken for the pulse to travel from x_2 to x_1 . Note that this complication is *not* relativistic in nature: this is the ultimate cause of the Doppler effect and is also present in the classical treatment. This transit time is equal to the difference $x_2 - x_1$ divided by the speed of the pulse as the K observer sees it. If the pulse moves at speed $-u'$ in K' (negative because it moves in the negative x -direction, towards the K observer at O), then the speed of the pulse moving *towards* the observer at O , as K sees it, is:

$$-u = \frac{-u' + v}{1 + (-u') \frac{v}{c^2}}$$

using the Lorentz equation for the velocities, above. Thus, the period between the pulses that the observer in K measures is:

$$\tau = t_2 - t_1 + \left(\gamma \frac{v}{f'} \right) \left(\frac{u' - v}{1 - \frac{u'v}{c^2}} \right)^{-1} = \frac{\gamma}{f'} + \frac{\gamma}{f'} \frac{v}{u' - v} \left(1 - \frac{vu'}{c^2} \right)$$

Replacing τ with $1/f$ and simplifying, we get the required result that gives the relativistic Doppler shift of *any* moving wave in terms of the stationary frequency, f' :

$$f = \gamma \left(1 - \frac{v}{u'} \right) f'$$

Ignoring the relativistic effects by taking $v \ll c$ or $c \rightarrow \infty$ (equivalent to $\gamma \rightarrow 1$) gives the *classical Doppler formula*:

$$f = \left(1 - \frac{v}{u'} \right) f'$$

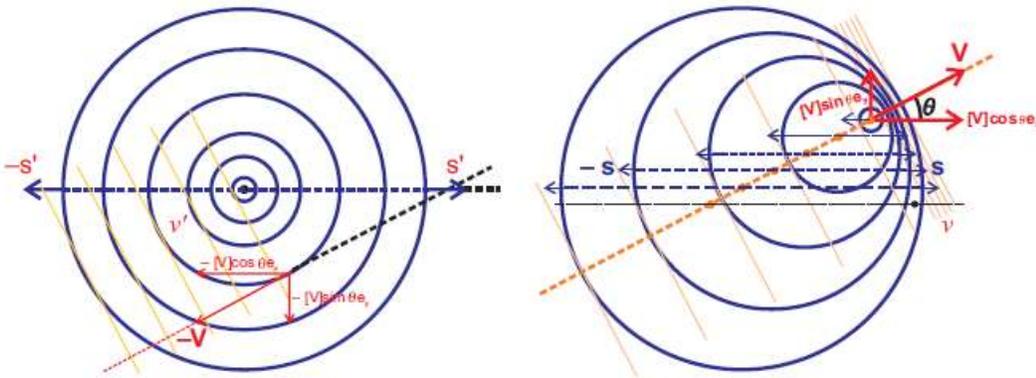
For electromagnetic radiation where $u' = c$ the formula becomes

$$f = \gamma \left(1 - \frac{v}{c} \right) f' = \gamma (1 - \beta) f' = f' \sqrt{\frac{1 - \beta}{1 + \beta}}$$

or in terms of wavelength:

$$\lambda = \lambda' \sqrt{\frac{1+\beta}{1-\beta}}$$

where λ' is the wavelength of the source at the origin O' as the observer in K' sees it.



For electromagnetic radiation, the limit to classical mechanics, $c \rightarrow \infty$, is instructive. The Doppler effect formula simply becomes $f = f'$. This is the correct result for classical mechanics, although it is clearly in disagreement with experiment. It is correct since classical mechanics regards the maximum speed of interaction — for electrodynamics, the speed of light — to be infinite. The Doppler effect, classical or relativistic, occurs because the wave source has time to move by the time that previous waves encounter the observer. This means that the subsequent waves are emitted further away (or closer) to the observer than they otherwise would be if the source were not in motion. The effect of this is to stretch (or compress) the wavelength of the wave as the observer encounters them. If however the waves travel instantaneously, the fact that the source is further away (or closer) makes no difference because the waves arrive at the observer no later or earlier than they would anyway since they arrive instantaneously. Thus, classical mechanics predicts that there should be no Doppler effect for light waves, whereas the relativistic theory gives the correct answer, as confirmed by experiment.

Summary

- 1. Situation in physics at the end of nineteenth century was described. The crisis faced by physics due to the problem of velocity of light was explained.*
- 2. The Michelson – Morley experiment was described in a nut shell. The null result of the experiment led to paradoxical situation, which was resolved by Einstein in his proposed theory of relativity.*
- 3. Postulates of the theory of relativity were described. This leads to a complete paradigm shift in the description of space-time.*
- 4. Lorentz transformations of the coordinates of events as viewed in moving frames of reference which replace the Galilean transformation of the Newtonian relativity were “derived”.*
- 5. The revolutionary and counter-intuitive implications of the consequences of Lorentz transformations – length contraction and time dilatation are discussed.*
- 6. The relativistic Doppler effect is derived.*