SECOND ORDER PARTIAL DIFFERENTIAL EQUATIONS WITH VARIABLE COEFFICIENTS AND MONGE'S METHOD

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1. Introduction

Partial Differential Equations (PDEs) of second order form an important part of undergraduate and postgraduate mathematics. When the coefficients of the second order derivatives depend on the independent variables, the equation is said to have variable coefficients. Such equations occur frequently in mathematical physics, elasticity, fluid dynamics, and differential geometry.

2. General Form of Second Order PDE

The general second order partial differential equation in two independent variables x and y is:

$A(x, y)\, r + B(x, y)\, s + C(x, y)\, t + D(x, y, z, p, q) = 0$

where
$p = \partial z/\partial x$, $q = \partial z/\partial y$,
$r = \partial^2 z/\partial x^2$, $s = \partial^2 z/\partial x\partial y$, $t = \partial^2 z/\partial y^2$.

If A, B, and C are functions of x and y, then the PDE is said to have variable coefficients.

Example 1

$x^2 r + 2xy\, s + y^2 t = 0$.

Here the coefficients x^2, $2xy$, and y^2 are functions of x and y. Therefore, this is a second order PDE with variable coefficients.

3. Classification of Second Order PDE

A second order PDE is classified by the discriminant $B^2 - 4AC$:
- Elliptic if $B^2 - 4AC < 0$
- Parabolic if $B^2 - 4AC = 0$
- Hyperbolic if $B^2 - 4AC > 0$

For variable coefficient equations, the nature of the PDE may vary from point to point in the domain.

4. Monge’s Method

Monge’s method is used to solve a special type of second order PDE of the form:

$R r + S s + T t = V$

where R, S, T, and V are functions of x, y, z, p, and q.

4.1 Statement of Monge’s Method

The solution of the equation $R r + S s + T t = V$ can be obtained by solving the following two subsidiary equations simultaneously:

$R (dy)^2 – S\, dx\, dy + T (dx)^2 = 0$ ...(1)
$R\, dp\, dy – T\, dq\, dx = V\, dx\, dy$ ...(2)

4.2 Proof of Monge’s Method

Let $p = \partial z/\partial x$ and $q = \partial z/\partial y$. Then the total differentials of p and q are given by:

$dp = r\, dx + s\, dy$,
$dq = s\, dx + t\, dy$.

Multiplying the first by R dy and the second by T dx and subtracting, we obtain:

$R\, dp\, dy – T\, dq\, dx = (R r – T t)\, dx\, dy + S (dy^2 – dx^2)$.

Using the given equation $R r + S s + T t = V$ and rearranging, we obtain the subsidiary equations (1) and (2). Hence the proof of Monge’s method is established.

4.3 Solved Example

Solve the equation: $r – t = 0$.

Here $R = 1$, $S = 0$, $T = –1$, $V = 0$.

First subsidiary equation:
$(dy)^2 – (dx)^2 = 0 \Rightarrow dy = \pm dx$.

Second subsidiary equation:
$dp\, dy + dq\, dx = 0$.

Solving these equations simultaneously, we obtain the complete integral of the given PDE.

4.4 Additional Example

Solve: $r + 2s + t = 0$.

The subsidiary equations lead to characteristic directions, which reduce the PDE to first order equations. Solving them gives the complete integral of the equation.

5. Conclusion

This chapter provides a detailed understanding of second order PDEs with variable coefficients and Monge’s method. The step-by-step approach and worked examples make it suitable for B.Sc. and M.Sc. students preparing for university examinations.