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B. Sc Part-III, Paper-V

Que (I) :- State and Prove Logarithmic Test.

Theorem :- Suppose that $u_n > 0$ and that $\lim \left(n \log \frac{u_n}{u_{n+1}} \right) = K$

Then the Series is Convergent $K > 1$ and divergent if $K < 1$

Proof :- let us Compare the given Series $\sum u_n$ with the auxiliary Series

$$\sum v_n = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \cdots + \frac{1}{n^p} + \cdots$$

whose nth term $v_n = \frac{1}{n^p}$. We Know that the Series $\sum v_n$ is Convergent if $p > 1$ and is divergent if $p \le 1$

$$\text{Now, } \frac{v_n}{v_{n+1}} = \frac{1}{n^p} \Big/ \frac{1}{(n+1)^p}$$

$$= \frac{(n+1)^p}{n^p} = \left(\frac{n+1}{n} \right)^p = \left(1 + \frac{1}{n} \right)^p$$

Case I :- Suppose that $\sum v_n$ is Convergent and hence $p > 1$ then by the Comparison test $\sum u_n$ is Convergent if

$$\frac{u_n}{u_{n+1}} > \frac{v_n}{v_{n+1}}$$

$$\text{i.e if } \frac{u_n}{u_{n+1}} > \left(1 + \frac{1}{n} \right)^p$$

$$\text{i.e if } \log \frac{u_n}{u_{n+1}} > p \log \left(1 + \frac{1}{n} \right)$$

$$\text{i.e if } \log \frac{u_n}{u_{n+1}} > p \left(\frac{1}{n} - \frac{1}{2n^2} + \cdots \right)$$

$$\text{i.e if } n \log \frac{u_n}{u_{n+1}} > p - \frac{p}{2n} + \cdots$$

$$\text{i.e if } \lim \left(n \log \frac{u_n}{u_{n+1}} \right) \ge p (>1)$$

$$\text{i.e if } \lim \left(n \log \frac{u_n}{u_{n+1}} \right) > 1$$

Case II :- Suppose that $\sum v_n$ is divergent and hence $p < 1$ then $\sum u_n$ is divergent if $\frac{u_n}{u_{n+1}} < \frac{v_n}{v_{n+1}}$

i.e if $Lt\left(n \log \frac{u_n}{u_{n+1}}\right) \leq p < 1$ as shown before.

Hence the theorem.

Que ② :— Theorem :— Evaluate :- $\int_0^{\pi/2} \frac{\log(1+\cos\alpha\cos x)}{\cos x} \cdot dx$

Proof :- Let $I = \int_0^{\pi/2} \frac{\log(1+\cos\alpha \cdot \cos x)}{\cos x} \cdot dx$ — ①

Diff. w.r.to α, we get

$$\frac{dI}{d\alpha} = \int_0^{\pi/2} \frac{(1-\sin\alpha \cdot \cos x)}{(1+\cos\alpha \cdot \cos x)} \cdot dx$$

$$= \int_0^{\pi/2} \frac{-\sin\alpha}{(1+\cos\alpha \cdot \cos x)} \cdot dx$$

But $\int \frac{dx}{a+b\cos x} = \frac{2}{\sqrt{a^2-b^2}} \tan^{-1}\left[\sqrt{\frac{a-b}{a+b}} \tan x/2\right]$ $a>b$

therefore $\frac{dI}{d\alpha} = -\sin\alpha \left[\frac{2}{\sqrt{1-\cos^2\alpha}} \tan^{-1}\left[\sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}} \tan x/2\right]\right]_0^{\pi/2}$

$$= -2\tan^{-1}\left(\tan \alpha/2\right) = -\alpha$$

$$\Rightarrow dI = -\alpha\, d\alpha$$

integrating $I = -\int \alpha \cdot d\alpha = \frac{\alpha^2}{2} + A$ — ②

Putting $\alpha = \frac{\pi}{2}$ in ① we get $I = 0$

From ② $0 - \frac{\pi^2}{8} + A$

$$\Rightarrow A = \frac{\pi^2}{8}$$

Therefore we get $I = \frac{1}{2}\left(\frac{\pi^2}{4} - \alpha^2\right)$