

Complex Analysis

Dr. Anupama Sinha

Department of Mathematics

H.D. Jain College, Ara

1. Schwarz Lemma

Statement: Let f be analytic in the unit disk $D = \{z : |z| < 1\}$, with $f(0) = 0$ and $|f(z)| \leq 1$ for all z in D . Then: (i) $|f(z)| \leq |z|$ for all z in D . (ii) $|f'(0)| \leq 1$. (iii) If equality holds for some non-zero z or $|f'(0)| = 1$, then $f(z) = e^{i\theta}z$ for some real θ .

Proof: Consider $g(z) = f(z)/z$ for $z \neq 0$ and define $g(0) = f'(0)$. Then g is analytic in D . Since $|f(z)| \leq 1$ and $f(0)=0$, we have $|g(z)| \leq 1/|z| * |f(z)| \leq 1$. By Maximum Modulus Principle, $|g(z)| \leq 1$ in D . Hence $|f(z)| \leq |z|$ and $|f'(0)| \leq 1$. If equality holds, Maximum Modulus Principle implies g is constant, hence $f(z) = e^{i\theta}z$.

Example: Let $f(z) = z/2$. Clearly analytic in $|z|<1$, $f(0)=0$ and $|f(z)| \leq 1$. Here $|f'(0)| = 1/2 \leq 1$, satisfying Schwarz Lemma.

2. Laurent Series

Statement: If f is analytic in an annulus $R_1 < |z - a| < R_2$, then $f(z)$ can be expressed as $f(z) = \sum_{n=-\infty}^{\infty} c_n (z-a)^n$.

Proof: Using Cauchy Integral Formula on two circles within the annulus, we split the integral into interior and exterior parts. Expanding the kernel $1/(\zeta - z)$ appropriately gives positive and negative powers. Collecting coefficients yields the Laurent expansion.

Example: Expand $f(z) = 1/(z(1-z))$ about $z=0$. We write $1/(z(1-z)) = 1/z * 1/(1-z)$. For $|z| < 1$: $1/(1-z) = \sum z^n$. Thus $f(z) = \sum z^{n-1}$, giving both negative and positive powers.

3. Isolated Singularities

Definition: A point a is an isolated singularity of f if f is analytic in $0 < |z-a| < r$. Types: (i) Removable: principal part zero. (ii) Pole of order m : finite principal part. (iii) Essential: infinitely many negative powers.

Examples: 1. $f(z)=\sin z / z$ at $z=0$ is removable. 2. $f(z)=1/z^2$ has pole of order 2 at $z=0$. 3. $f(z)=e^{1/z}$ has essential singularity at $z=0$.

4. Meromorphic Functions

Definition: A function is meromorphic in a domain D if it is analytic in D except at isolated poles.

Theorem: A meromorphic function can be written locally as ratio of two analytic functions. Proof:

Near a pole a , write Laurent expansion. Since principal part finite, multiply by suitable power $(z-a)^m$ to remove singularity, giving analytic numerator and denominator.

Example: $f(z)=1/\sin z$ is meromorphic in \mathbb{C} since zeros of $\sin z$ give poles of f .