Theorems on cyclic Groups.

Theorem:— Every cyclic group is necessarily abelian.

Proof:— Let G be cyclic group generated by a so that $G = \{a\}$.

The group G is abelian.

Let $x, y \in G$ be arbitrary so that $\exists\, m, n \in \mathbb{Z}$, $x = a^n$; $y = a^m$

$$\therefore \quad xy = a^n \cdot a^m = a^{n+m} = a^{m+n}$$

$$= a^m \cdot a^n = yx.$$

(For $(\mathbb{Z}, +)$ is an abelian group)

$$\therefore \quad xy = yx \quad \forall\, x, y \in G.$$

This proves that G is abelian.

Hence the proposition.

Theorem:— If a is a generater of a cyclic group G, then a^{-1} is also a generater of G.

OR.

A cyclic group has two generaters.

Proof:— Let a cyclic group G be generated by a so that $G = \{a\}$ and a is a generator of G.

Let $x \in G$ be arbitrary, then $\exists$ an integer n such that $x = a^n = (a^{-1})^{-n}$, where $-n$ is also an integer.

This proves that the elements of G are expressible as some integral powers of a^{-1}

i.e. G is generated by a^{-1},

Consequently a^{-1} is a generator of G.

Proved.

Problem:-01. How many generators are there of the cyclic group of order 10?

Solution:— Let G be a cyclic group of order 10 generated by an element a. Then $O(a) = O(G) = 10$

Evidently $G = \{a, a^2, a^3, a^4, a^5, a^6, a^7, a^8, a^9, a^{10} = e\}$.

If the H.C.F of m and n is d, then we write $(m,n) = d$.

An element $a^m \in G$ is also a generator of G if $(m, 10) = 1$.

Thus there are four generators of G namely,

a, a^3, a^7, a^9. Ans.