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Fundamental Theorem of Algebra

Statement: Every non-constant polynomial with complex coefficients has at least one complex root. Equivalently, every polynomial of degree $n \geq 1$ has exactly n roots in the complex plane counting multiplicities.

Proof (Using Liouville's Theorem): Let $p(z) = a_n z^n + \dots + a_0$ be a non-constant polynomial. Assume $p(z)$ has no zero in \mathbb{C} . Then $1/p(z)$ is entire. As $|z| \rightarrow \infty$, $|p(z)| \rightarrow \infty$, so $|1/p(z)| \rightarrow 0$. Hence $1/p(z)$ is bounded and entire. By Liouville's Theorem, $1/p(z)$ is constant. Therefore $p(z)$ is constant, which contradicts the assumption. Hence $p(z)$ must have at least one complex root. By factoring repeatedly, we obtain exactly n roots counting multiplicity.

Examples: 1. Polynomial $z^2 + 1 = 0$ Roots: $z = i, -i$. 2. Polynomial $z^3 - 1 = 0$ Roots: $1, e^{2\pi i/3}, e^{4\pi i/3}$. 3. Polynomial $z^2 + 4z + 5 = 0$ Roots: $-2 \pm i$.

Power Series

Definition: A power series centered at a is of the form $\sum_{n=0}^{\infty} c_n (z - a)^n$. There exists $R \geq 0$ called the radius of convergence such that: - The series converges if $|z - a| < R$, - Diverges if $|z - a| > R$.

Radius of Convergence: Using Cauchy-Hadamard formula: $1/R = \limsup |c_n|^{1/n}$.

Examples: 1. Geometric Series: $\sum z^n = 1/(1 - z)$, $|z| < 1$. Radius of convergence $R = 1$. 2. Exponential Function: $e^z = \sum z^n / n!$ Converges for all $z \in \mathbb{C}$. Radius of convergence $R = \infty$. 3. Series $\sum n z^n$: Using ratio test, $R = 1$. 4. Function $1/(1 - z)^2 = \sum (n+1) z^n$, $|z| < 1$.

Important Properties: - Power series are analytic inside their radius of convergence. - They can be differentiated and integrated term-by-term within $|z - a| < R$. - If two power series are equal in a domain, then their coefficients are equal (Uniqueness Theorem).