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Partial Differential Equations (Semester IV)

Monge's Method and Classification of Second Order Linear Equations

1. Monge's Method

Monge's method is used to solve second order partial differential equations of the form: $R r + S s + T t = V$, where $r = \partial^2 z / \partial x^2$, $s = \partial^2 z / \partial x \partial y$, $t = \partial^2 z / \partial y^2$, and R, S, T, V are functions of x, y, z, p, q where $p = \partial z / \partial x$, $q = \partial z / \partial y$. Step 1: Form Monge's auxiliary equations: $R (dy)^2 - S dx dy + T (dx)^2 = 0$ Step 2: Solve this quadratic equation to obtain two characteristic directions. Step 3: Using these directions, form subsidiary equations: $R dp dy + T dq dx - V dx dy = 0$ Step 4: Integrate to obtain intermediate integrals.

Proof (Outline of Monge's Method)

Consider the PDE $Rr + Ss + Tt = V$. Using total differential relations: $dp = r dx + s dy$, $dq = s dx + t dy$. Multiply first by $R dy$ and second by $T dx$ and eliminate s . After simplification, we obtain: $R (dy)^2 - S dx dy + T (dx)^2 = 0$. This quadratic form determines characteristic curves. Solving along these curves reduces the PDE to ordinary differential equations.

Example 1

Solve: $r + t = 0$. Here $R = 1, S = 0, T = 1$. Auxiliary equation: $(dy)^2 + (dx)^2 = 0$. Thus $dy/dx = \pm i$. Hence the solution becomes: $z = \phi(x + iy) + \psi(x - iy)$. This represents harmonic functions.

Example 2

Solve: $r - t = 0$. Here $R = 1, S = 0, T = -1$. Auxiliary equation: $(dy)^2 - (dx)^2 = 0$. So $dy/dx = \pm 1$. Hence solution: $z = \phi(x + y) + \psi(x - y)$.

2. Classification of Second Order Linear PDE

Consider general second order linear PDE: $A u_{xx} + B u_{xy} + C u_{yy} + \text{lower order terms} = 0$.
Discriminant: $D = B^2 - 4AC$. 1. Hyperbolic if $D > 0$ 2. Parabolic if $D = 0$ 3. Elliptic if $D < 0$

Hyperbolic Example

Equation: $u_{xx} - u_{yy} = 0$. Here $A = 1$, $B = 0$, $C = -1$. $D = 0^2 - 4(1)(-1) = 4 > 0$. Hence hyperbolic.
Example from physics: Wave equation.

Parabolic Example

Equation: $u_{xx} = 0$. Here $A = 1$, $B = 0$, $C = 0$. $D = 0$. Hence parabolic. Example: Heat equation in one direction.

Elliptic Example

Equation: $u_{xx} + u_{yy} = 0$. Here $A = 1$, $B = 0$, $C = 1$. $D = 0 - 4 = -4 < 0$. Hence elliptic. Example: Laplace equation.

Conclusion: Monge's method reduces certain second order PDEs to first order systems along characteristic curves. Classification helps determine nature of solutions and physical behavior of equations.