# Alternating Group.

The set of $\frac{n!}{2}$ even permutations of degree $n$ is called alternating set and is denoted by $A_n$. The set $A_n$ of $\frac{n!}{2}$ even permutations from a finite non-abelian group relative to permutation multiplication as composition. This group is called alternating group of order $\frac{n!}{2}$.

**Theorem** – The set $P_n$ of all permutations on $n$ sybols is a finite non-abelian group of order $n!$ w.r.t. composition of mappings as the operation.

**Proof:** — Let $S = \{a_1, a_2, --, a_n\}$ be a set having $n$ distinct objects. Let $P_n$ be the set of all permutations on $n$ symbols belonging to the set $S$.

(i) closure property. $\forall f, g \in P_n \Rightarrow fg \in P_n$.

For $\forall f, g \in P_n \Rightarrow f \begin{pmatrix} a_1 & a_2 & -- & a_n \\ b_1 & b_2 & -- & b_n \end{pmatrix}$,

$$g = \begin{pmatrix} b_1 & b_2 & -- & b_n \\ c_1 & c_2 & -- & c_n \end{pmatrix}$$

$$\Rightarrow fg = \begin{pmatrix} a_1 & a_2 & -- & a_n \\ c_1 & c_2 & -- & c_n \end{pmatrix}$$

$$\Rightarrow fg \in P_n.$$

Where $b_1, b_2, --, b_n$ or $c_1, c_2, --, c_n$ are nothing but an arrangement of the same $n$ elements $a_1, a_2, --, a_n$.

(ii) $o(P_n) = n!$ for the elements of $S$ can be arranged in $n!$ different ways and hence $P_n$ contains $n!$ distinct permutations of degree $n$.

(iii) Associativity, ~~f(gh) = f~~ $(fg)h = f(gh)$ $\forall$ $f, g, h \in P_n$.

$f, g, h \in P_n \Rightarrow f, g, h$ are expressible as

$$f = \begin{pmatrix} a_1 & a_2 & -- & a_n \\ b_1 & b_2 & -- & b_n \end{pmatrix}, \quad g = \begin{pmatrix} b_1, & b_2, & -- & b_n \\ c_1, & c_2, & -- & c_n \end{pmatrix}$$

$$h = \begin{pmatrix} c_1 & c_2 & -- & c_n \\ d_1 & d_2 & -- & d_n \end{pmatrix}.$$

Where the elements $b_1, b_2, \ldots, b_n$. $c_1, c_2, \ldots, c_n$, $d_1, d_2, \ldots, d_n$ are simply different arrangements of the same $n$ elements $a_1, a_2, \ldots, a_n$.

Now $(fg)h = \begin{pmatrix} a_1, a_2, \ldots a_n \\ c_1, c_2, \ldots c_n \end{pmatrix} \begin{pmatrix} c_1 c_2 \ldots c_n \\ d_1 d_2 \ldots d_n \end{pmatrix}$

$$= \begin{pmatrix} a_1, a_2 \ldots a_n \\ d_1, d_2 \ldots d_n \end{pmatrix} \quad \text{---} (1)$$

$f(gh) = \begin{pmatrix} a_1 a_2 \ldots a_n \\ b_1 b_2 \ldots b_n \end{pmatrix} \begin{pmatrix} b_1 b_2 \ldots b_n \\ d_1 d_2 \ldots d_n \end{pmatrix}$

$$= \begin{pmatrix} a_1 a_2 \ldots a_n \\ d_1 d_2 \ldots d_n \end{pmatrix} \quad \text{---} (2)$$

Equating eqn (1) and (2), we get the required result.

(IV) Existence of identity.

Let $I = \begin{pmatrix} a_1 & a_2 \ldots a_n \\ a_1 & a_2 \ldots a_n \end{pmatrix} = \begin{pmatrix} b_1 b_2 \ldots b_n \\ b_1 b_2 \ldots b_n \end{pmatrix}$

Where the elements $b_1, b_2, \ldots b_n$ are simply different arrangement of the same $n$ elements $a_1, a_2 \ldots a_n$. Clearly $I$ is a permutation of degree $n$.

$fI \begin{pmatrix} a_1 \ldots a_n \\ b_1 \ldots b_n \end{pmatrix} \begin{pmatrix} b_1 \ldots b_n \\ b_1 \ldots b_n \end{pmatrix} = \begin{pmatrix} a_1 \ldots a_n \\ b_1 \ldots b_n \end{pmatrix} = f$

$If = \begin{pmatrix} a_1 \ldots a_n \\ a_1 \ldots a_n \end{pmatrix} \begin{pmatrix} a_1 \ldots a_n \\ b_1 \ldots b_n \end{pmatrix} = \begin{pmatrix} a_1 \ldots a_n \\ b_1 \ldots b_n \end{pmatrix} = f$

$\therefore fI = If = f$.

This shows that $I$ is an identity element of $P_n$ and $I \in P_n$.

(V) Existence of Inverse.

Let $f \in P_n \Rightarrow f = \begin{pmatrix} a_1 \ldots a_n \\ b_1 \ldots b_n \end{pmatrix}$

take $f^{-1} = \begin{pmatrix} b_1 \ldots b_n \\ a_1 \ldots a_n \end{pmatrix}$

Then $ff^{-1} = \begin{pmatrix} a_1 \ldots a_n \\ b_1 \ldots b_n \end{pmatrix} \begin{pmatrix} b_1 \ldots b_n \\ a_1 \ldots a_n \end{pmatrix} = \begin{pmatrix} a_1 \ldots a_n \\ a_1 \ldots a_n \end{pmatrix} = I$

$$f^{-1}f = \begin{pmatrix} b_1 & \cdots & b_n \\ a_1 & \cdots & a_n \end{pmatrix} \begin{pmatrix} a_1 & \cdots & a_n \\ b_1 & \cdots & b_n \end{pmatrix}$$

$$= \begin{pmatrix} b_1 & \cdots & b_n \\ b_1 & \cdots & b_n \end{pmatrix} = I. \text{ Thus } ff^{-1} = f^{-1}f = I.$$

This proves that the inverse of an arbitrary element $f \in P_n$ is $f^{-1} \in P_n$.

Commutative law:— Multiplication of permutations is not commutative. The proof of this is illustrated by an example.

If $f = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$, $g = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$, then

$$fg = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \begin{pmatrix} 2 & 3 & 1 \\ 1 & 3 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} \quad \text{——— (1)}$$

$$gf = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \begin{pmatrix} 2 & 1 & 3 \\ 3 & 2 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} \quad \text{——— (2)}$$

By (1) and (2), $fg \neq gf$.

This proves the required result.