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RESIDUES AND CAUCHY RESIDUE THEOREM

1. Introduction to Residues
Let $f(z)$ be analytic in a region except at isolated singularities. If $z = a$ is an isolated singularity, then the Laurent expansion of $f(z)$ about $z = a$ is: $f(z) = \ldots + b_{-2}/(z-a)^2 + b_{-1}/(z-a) + b_0 + b_1(z-a) + \ldots$
The coefficient b_{-1} is called the **residue** of $f(z)$ at $z = a$.

2. Methods of Finding Residues
(i) If a is a simple pole, then:
$\text{Res}[f(z), a] = \lim_{z\to a} (z-a)f(z)$.

(ii) If a is a pole of order m:
$\text{Res}[f(z), a] = \frac{1}{(m-1)!} \lim_{z\to a} \frac{d^{m-1}}{dz^{m-1}} [(z-a)^m f(z)]$.

3. Cauchy Residue Theorem (Statement)
If $f(z)$ is analytic inside and on a simple closed contour C except at finite isolated singularities a1, a2, ..., an inside C, then:
■_C $f(z)\, dz = 2\pi i \sum \text{Res}[f(z), a_k]$

Proof (Sketch)
Let small circles be drawn around each singularity. Using Cauchy's Integral Theorem and deformation of contour, the integral reduces to sum of integrals around small circles. Using Laurent expansion, only the coefficient of $1/(z-a)$ contributes, giving $2\pi i$ times the residue. Hence proved.

4. Examples

Example 1: Find Res at $z=1$ of $f(z)=1/(z-1)$.
Solution: Simple pole $\to$ residue = 1.

Example 2: Find Res at $z=0$ of $f(z)=1/z^2$.
Solution: No $1/z$ term $\to$ residue = 0.

Example 3: Evaluate ■ $|z|=2$ $dz/(z-1)$.
Solution: Residue at $z=1$ is 1 $\to$ Integral = $2\pi i$.

Example 4: Evaluate ■ $|z|=3$ $z/(z^2+1)\, dz$.
Poles at $\pm i$ inside contour.
Res at $i = 1/(2i)$, Res at $-i = -1/(2i)$.
Sum = 0 $\to$ Integral = 0.

Example 5: Evaluate ■ $|z|=2$ $e^z/z\, dz$.
Residue at 0 equals coefficient of $1/z$ in expansion of $e^z/z = 1/z + 1 + \ldots$
Residue = 1 $\to$ Integral = $2\pi i$.

Example 6 (Real Integral):
Evaluate $\int_{-\infty}^{\infty} dx/(x^2+1)$.

Using contour integration in upper half plane, pole at i gives residue 1/(2i).
Integral = π.

5. Evaluation of Real Integrals Using Residue Theorem

Steps:

1. Extend integral to complex plane.
2. Choose suitable contour.
3. Identify singularities inside contour.
4. Compute residues.
5. Apply Cauchy Residue Theorem.
6. Take limit as radius $\to \infty$.