A’C + AC’D + AB’C’ + ABD = A’C + AB’C’ + ABD

(5) Interchange Theorem

$$A B + A' C = (A + C) (A' + B)$$

Proof: RHS $= (A + C) (A' + B)$

$= A A' + A B + A' C + B C$	[L7a]
$= 0 + A B + A' C + B C$	[L4a]
$= A B + A' C + B C$	[L3b]
$= A B + A' C = \text{LHS}$	[T4a]

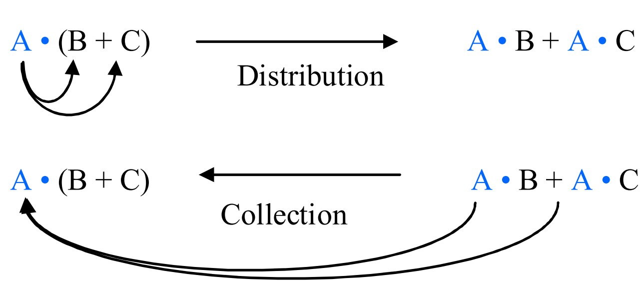
The interchange theorem is not for simplification of Boolean expressions. It is used for the conversion of an expression from a sum-of-products to a product-of-sums, or vice versa. In applying this theorem from the conversion between a SOP expression and a POS expression, a variable should appear in true form in one product and in complemented form in the other product. Or the true form of a variable in one sum and its completed form in another sum. If such a variable does not exist, the theorem is not applicable. The conversion involves a process of interchanging literals. If the variable appears in true and complemented forms is A. The literals associated with A and A’ in the sum-of-products (LHS of Theorem 5) are B and C respectively. To change the expression to a product-of-sums (RHS of Theorem 5), B will associate with A’ and C becomes a partner of A. The process of interchanging associating literals is also applicable to the conversion from product-of-sums expression to sum-of-products expression.

Conversions between Sum-of-Products and Product-of-Sums Expressions

When a product-of-sums expression is converted to a sum-of-product expression, the process is called “multiplying-out”. On the contrary, the change of a sum-of-products expression to a product-of-sums expression is called “factoring”.

The distributive laws are always used in conversions between sum-of-products and product-of-sums. In fact, there are two opposite processes in the distributive laws. These two processes are shown below. Distribution is to distribute a literal to each and every literal in another term. Collection is to collect a common literal from each and every term. One process is the reverse of the other process.

Distributive law (7a)



Distributive law (7b)

$A + B \cdot C \longrightarrow (A + B) \cdot (A + C)$

Distribution

$A + B \cdot C \longleftarrow (A + B) \cdot (A + C)$

Collection

When converting a product-of-sums expression to a sum-of-products expression, applying the distributive law (7a) is correct but may not be the best approach. For instance,

$$
\begin{aligned}
&(A + B + C)(A + B + D) \\
&= AA + AB + AD + AB + BB + BD + AC + BC + CD
\end{aligned}
$$

Not only that it is tedious in multiplying out the two sums, the result is not simple and needs to be simplified using the idempotency law and the absorption theorem.

$$
\begin{aligned}
&AA + AB + AD + AB + BB + BD + AC + BC + CD \\
&= A + AB + AD + B + BD + AC + BC + CD && [L2] \\
&= A + B + BD + BC + CD && [T2a] \\
&= A + B + CD && [T2a]
\end{aligned}
$$

If the second form of the distributive law (7b) is used, it requires only one step in getting the simplest sum-of-products expression.

$$
\begin{aligned}
&(A + B + C)(A + B + D) \\
&= [\, \underline{(A + B) +} C]\, [\, \underline{(A + B) +} D] \\
&= \underline{(A + B)} + CD
\end{aligned}
$$

❖ Example 3.7

The collection process of the distributive law can be applied to more than two terms. It is illustrated in this example by converting the following expression to a simplest sum-of-products expression.

$$(\underline{A + B} + C)(\underline{A + B} + D)(\underline{A + B} + E)(B' + D')$$

$$
\begin{aligned}
&(\underline{A + B} + C)(\underline{A + B} + D)(\underline{A + B} + E)(B' + D') \\
&= (\ \underline{A + B}\ + CDE\ )\ (B' + D')
\end{aligned}
$$

$$= [B + (A + CDE)] (B' + D')$$
$$= BD' + B'(A + CDE) \qquad [T5]$$
$$= BD' + AB' + B'CDE$$

❖ Example 3.8

The interchange theorem is employed in the conversion of (BCD' + B'D + AB) to a product-of-sums expression. B or B' appears in every product. Therefore the expression can be formulated into two parts, one with B and another with B'.

$$BCD' + B'D + AB = B(A + CD') + B'D$$

By applying the interchange theorem, the expression becomes

$$(B + D) (B' + A + CD')$$

By distributing A + B' to C and D'

$$A + B' + CD' = (A + B' + C) (A + B' + D')$$

Thus $\quad BCD' + B'D + A'B = (B + D) (A + B' + C) (A + B' + D')$

❖ Example 3.9

This example illustrates the conversion of a product-of-sums expression to a sum-of-products expression in its simplest form. For a simplest or minimal sum-of-products expression, the number of product terms should be a minimum. The total number of literals should also be a minimum. There should not be any expression in which the number of product terms or/and the total number of literal are smaller.

$$(A + B) (A' + C) (C' + D)$$
$$= (AC + A'B) (C' + D) \qquad [T5]$$
$$= ACC' + ACD + A'BC' + A'BD \qquad [L7a]$$
$$= ACD + A'BC' + A'BD$$

If the distributive law (7a) is used to multiply out (A + B)(A' + C), the result is (AC + A'B + BC) instead of (AC + A'B). The consensus theorem has to be used in the elimination of BC. Thus in the interchange theorem, the consensus theorem has already been built in.

❖ Example 3.10

Convert the sum-of-products expression (A'B + CD) to a product-of-sums. The distributive law (7b) is used to distribute A'B to C and D.

A'B + CD = (A'B + C) (A'B + D)

For each of the two expressions within the parentheses, C as well as D is again distributed to A'B using the same distributive law.

C + A'B = (C+ A') (C + B)

D + A'B = (D+ A') (D + B)

Thus A'B + CD = (A'+ C) (B + C) (A' + D) (B + D)

From the four examples given above for conversions between sum-of-products and product-of-sums expression, it is suggested that the collection process of the distributive law be used first and followed by the interchange theorem, and then the distribution process of the distributive law.

Examples 3.7 to 3.10 and the example before Example 3.7 provide a 3-step guideline that can be used for conversions between sum-of-products and product-of-sums expressions. Note that this guideline may not be the best approach in all cases. The procedure is as follows:

(i) Apply the collective process of the distributive law.
(ii) Apply the Interchange theorem.
(iii) Apply the distributive process of the distributive law.

Ignore any step that is not applicable. Basic laws and theorems that can be used for simplification must be used when they are applicable in any step.

(6) DeMorgan's theorem
(a) (A • B)' = A' + B'
(b) (A + B)' = A' • B'

Table 3.7 Proof of DeMorgan's theorem (6a).

A B	A B	Left-hand-side of (6a) (A B)'	A' B'	Right-hand-side of (6a) A' + B'	A'•B'
0 0	0	1	1 1	1	1
0 1	0	1	1 0	1	0
1 0	0	1	0 1	1	0
1 1	1	0	0 0	0	0

DeMorgan's theorem is used to manipulate the inversion or complement of a Boolean expression. The first form of the theorem (6a) is proved in Table 3.7 using the perfect induction method. The proof for the second form is left as an exercise. By comparing the two rightmost columns of Table 3.7, it shows that

$$(A\ B)' \neq A'\ B'$$

Similarly, $(A + B)' \neq A' + B'$

DeMorgan's theorem is not limited to just two variables and can be extended to any number of variables. The general forms are as follows.

(a) $(x_1 \bullet x_2 \bullet x_3 \bullet \ldots\ldots\ldots \bullet x_{n-1} \bullet x_n)' = x_1' + x_2' + x_3' + \ldots\ldots\ldots + x_{n-1}' + x_n'$
(b) $(x_1 + x_2 + x_3 + \ldots\ldots\ldots + x_{n-1} + x_n)' = x_1' \bullet x_2' \bullet x_3' + \ldots\ldots\ldots \bullet x_{n-1}' \bullet x_n'$

The general DeMorgan's theorem can be proved using the induction method. To prove the first form (a), the starting step is to show that the theorem is true for two variables, which has already been established. The second step is to show that it is also true for three variables.

$$(x_1 \bullet x_2 \bullet x_3)' = ((x_1 \bullet x_2) \bullet x_3)' = (x_1 \bullet x_2)' + x_3'$$
$$= (x_1' + x_2') + x_3' = x_1' + x_2' + x_3'$$

The first form of the 2-variable DeMorgan's theorem is applied to the second equality in the above proof. Now assume the general DeMorgan's theorem is true for (n-1) variables.

$$(x_1 \bullet x_2 \bullet x_3 \bullet \ldots\ldots\ldots \bullet x_{n-1})' = x_1' + x_2' + x_3' + \ldots\ldots\ldots + x_{n-1}'$$

It is necessary to prove that the theorem is also true for n variables.

$$(x_1 \bullet x_2 \bullet x_3 \bullet \ldots\ldots\ldots \bullet x_{n-1} \bullet x_n)'$$
$$= ((x_1 \bullet x_2 \bullet x_3 \bullet \ldots\ldots\ldots \bullet x_{n-1}) \bullet x_n)'$$
$$= (x_1 \bullet x_2 \bullet x_3 \bullet \ldots\ldots\ldots \bullet x_{n-1})' + x_n'$$
$$= (x_1' + x_2' + x_3' + \ldots\ldots\ldots + x_{n-1}') + x_n'$$
$$= x_1' + x_2' + x_3' + \ldots\ldots\ldots + x_{n-1}' + x_n'$$

The 2-variable DeMorgan's theorem is used for the second equality. DeMorgan's theorem for (n-1) variables is applied to the third equality. The second general form of DeMorgan's theorem for n variables can be proved in a similar manner.

❖ Example 3.11

The application of DeMorgan's theorem is illustrated by converting the expression [A' + B(C + D') + E]' to a sum-of-products form. In eliminating a prime outside a pair of parentheses or brackets, each term within the parentheses or brackets is complemented, and the logical operation AND is changed to OR, or vice versa.

$$[A' + B(C + D') + E]'$$
$$= A \bullet [B(C + D')]' \bullet E'$$
$$= A \bullet [B' + (C + D')'] \bullet E'$$
$$= A \bullet (B' + C'D) \bullet E'$$
$$= AB'E' + AC'DE'$$

3.5 Minimization of Literals

Under certain circumstances, the number of literals in a Boolean expression needs to be minimized. The expression with a minimum number of literals may not necessarily be a sum-of products or product-of-sums expression. Inversions can be applied only to variables if an expression is defined as one with a minimum number of literals. The collection process of the distributive law is useful in reducing the number of literals. The process may also be applied to part of an expression and the common factor is not limited to just a single literal.

❖ Example 3.12

$$F(A,B,C,D) = BD + CD + A'BC + ABC'$$

Given above is the simplest sum-of-products form for a 4-variable function F(A,B,C,D). By factoring D from the first two products and B from the last two products, the expression becomes

$$D(B + C) + B(A'C + AC')$$

The number of literal for the given expression can also be minimized by factoring different literals, which results in the following two expressions.

$$CD + B(D + A'C + AC')$$

and $$B(D + AC') + C(D + A'B)$$

Each of the three expressions has eight literals.

❖ Example 3.13

$$F(A,B,C,D) = (A + C')(B + D)(A' + C + D)$$